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THE THEORY AND APPLICATION OF LINEAR OPTIMAL CONTROL

EDMUND G. RYNASKI AND RICHARD F. WHITBECK

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EDMUND G. RYNASKI AND RICHARD F. WHITBECK

FOREWORD

The research documented in this report was performed for the Air Force Flight Dynamics Laboratory of the Research and Technology Division, Wright-Patterson Air Force Base, Ohio, by the Flight Research Department of the Cornell Aeronautical Laboratory, Inc., of Buffalo, New York. This study was done under Air Force Contract No. AF33(615)-1541, Project No. 8219 "Stability and Control Investigation", and Task No. 821904. The project was administered by Mr. Frank George of the Flight Dynamics Laboratory. The work was performed primarily by the principal investigator, Mr. Edmund Rynaski, and by Dr. Richard Whitbeck.

Important suggestions and guidance were provided by Mr. R. Anderson of the Flight Dynamics Laboratory, and by Mr. W. Deazley, Mr. J.N. Ball, and Mr. J. M. Schuler of Cornell. Grateful acknowledgement is made to Mr. W. Shed and Mr. C. Mesiah for computing services and Mrs. J. Martino and Miss D. Kantorski for their report preparation skills. The research documented in this report is based upon the original theories developed by Dr. R.E. Kalman and Dr. S.S.L. Chang.

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ABSTRACT

Linear optimal control theory has produced an important synthesis technique for the design of linear multivariable systems. In the present study, efficient design procedures, based on the general optimal theory, have been developed. These procedures make use of design techniques which are similar to the conventional methods of control system analysis. Specifically, a scalar expression is developed which relates the closed-loop poles of the multi-controller, multi-output optimal system to the weighting parameters of a quadratic performance index. Methods analogous to the root locus and Bode plot techniques are then developed for the systematic analysis of this expression. Examples using the aircraft longitudinal equations of motion to represent the object to be controlled are presented to illustrate design procedures which can be carried out in either the time or frequency domains. Both the model-in-the-performance-index and model-following concepts are employed in several of the examples to illustrate the model approach to optimal design.

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LIST OF SYMBOLS

Vectors

x — state vector; whose components define the variables of the first-order set of equations of motion of the plant

y — the output vector; defined by a transformation on x, the set appearing in the performance index

u — the control vector; the input to the plant

u_o — the optimal control vector; the input motion that forces the plant to respond optimally

λ — the adjoint state vector; the undetermined multiplier of the Euler-Lagrange equations

$x(s)$ — the Laplace transformed state vector

$y(s)$ — the Laplace transformed output vector

$u(s)$ — the Laplace transformed control vector

$u_o(s)$ — the Laplace transformed optimal control vector

$\lambda(s)$ — the Laplace transformed adjoint state vector

$a(t)$ — generalized deterministic disturbance vector

$a(s)$ — the Laplace transformed disturbance vector

$\mathfrak{z}(s)$ — vector defined to be analytic in the left-half plane

x_m — model state variable

η — fictitious state vector associated with the model in the performance index; $\dot{\eta} = L\eta$

$\xi(s)$ — a vector consisting of polynomial entries which define the numerators of the optimal control

Matrices

F — n x n matrix; matrix of constants that define the interactions among the state variables of the plant

G — n x p input matrix; matrix of constants defining the effect of a control on the state rates

H — r x n matrix transformation on x that defines the output, y = Hx

Matrices (continued)

$\mathcal{Q}$	r x r matrix of constants that weight the output in the performance index
$\mathcal{R}$	p x p matrix of constants weighting the control in the performance index
L	the model system matrix; matrix of constants defining the interactions among the model state variables
M	matrix of constants weighting the output rates in the performance index
$\mathcal{S}$	matrix of numbers weighting the derivative of the state in the performance index
T	matrix of constants weighting the control rates in the performance index; collinear transformation $x = Ty$ to transform the state vector to a set of uncoupled equations
P	variable of the Riccati equation
K	matrix of feedback gains, $K = R^{-1}G'P$
I	identity matrix
Λ	modal matrix
$[Is-F]^{-1}$	transition matrix of the plant
$\frac{Y}{U}(s)$	= W(s) = matrix of transfer functions relating the outputs $y_i(s)$ to the inputs $u_j(s)$
$B(s)$	$= H[Is-F]^{-1}$ = transition matrix of the output

Matrix and Vector Notation and Abbreviations

$[\ ]$	denotes a matrix or a vector
$\|\ \|$	denotes a determinant
$[\ ]'$	transpose of a vector or a matrix
$[\ ]^{-1}$	inverse of a matrix
$[\ ]^{adj}$	adjugate of a matrix defined by $[A]^{-1} = [A]^{adj}/\|A\|$
$\overline{[\ ]}$	conjugate of a polynomial matrix of s, $W(-s) = \overline{W}(s)$
$[\ ]_*$	conjugate transpose of a polynomial matrix of s, $W_*(s) = W'(-s) = [\overline{W}(s)]'$

Matrix and Vector Notation and Abbreviations (Continued)

$[\;]_{ij}$	the matrix element appearing in the i^{th} row and j^{th} column
$x(0)$	initial conditions of a vector
$[\;]_i$	the i^{th} component of a vector
lhp	left-hand plane
rhp	right-half plane
min	minimum
lim	limit
sup	supremum – least upper bound
opt	optimum
dim	dimension

Determinants and Scalars

$D(s)$	characteristic polynomial of the open-loop plant, $D(s) = \lvert Is - F \rvert$
$\Delta(s)$	characteristic polynomial of the optimal system, $\Delta(s) = \lvert Is - F + GK \rvert$
A_{ij}	first minor obtained by deleting the i^{th} row and j^{th} column of $\lvert Is - F \rvert$
$\bar{D}(s)$	$\bar{D}(s) = D(-s)$
$A_{ij,kl}$	second minor obtained by deleting the i^{th} and k^{th} rows and the j^{th} and l^{th} columns of $\lvert Is - F \rvert$
s	$= \sigma + j\omega$, Laplace transform variable
V	the performance index, $2V = \int_0^\infty (y'Qy + u'Ru)\,dt$
η	the integrand of the performance index, $\eta = \frac{1}{2}(y'Qy + u'Ru)$
$\Gamma(s)$	left-half plane poles of $W_x QB[x(0) + a(s)]$ excluding the open-loop poles, $D(s)$
$\mathcal{L}$	the Lagrangian function; $\mathcal{L} = \frac{1}{2}\eta + \lambda'\cdot(-\dot{x} + Fx + Gu)$
$\mathcal{H}$	the Hamiltonian function, $\mathcal{H} = \frac{1}{2}\eta + \lambda'\cdot(Fx + Gu)$

Aircraft Notation

$\Delta\dot{\theta}$	incremental pitching velocity, deg/sec
$\Delta\theta$	incremental pitch angle, deg
$\Delta\alpha$	incremental angle of attack, deg
ΔV	incremental change in airspeed, ft/sec

Aircraft Notation (Continued)

$\Delta\delta_E$	incremental elevator deflection, deg
$\Delta\delta_T$	incremental change in thrust, lb
ω_n	short period natural frequency, rad/sec
ζ	short period damping ratio
ω_p	phugoid natural frequency, rad/sec
ζ_p	phugoid damping ratio

SECTION 1
INTRODUCTION

This report describes the results of a study of the characteristics of linear optimal control. As used in this report, a system describable by a set of constant-coefficient linear differential equations of motion is said to be linear. Optimal control is a technique for control system synthesis by which unique control input motions are specified that minimize a functional of the motions of the system. This functional is called a performance index. Linear optimal control is an optimal control synthesis procedure for linear systems whereby the control motions are uniquely determined by a feedback law consisting of a constant linear sum of the variables, or states of the system.

Mathematically, the problem can be defined as follows:

For some initial condition of the state, $x(0)$, find the control u_0 that minimizes the quadratic performance index

$$2V = \int_0^{\infty} (y'Qy + u'Ru)dt$$

subject to the natural constraint of the linear, constant coefficient equations of motion of the plant written in the first-order form

$$\dot{x} = Fx + Gu \qquad y = Hx$$

The performance index is a scalar quantity consisting of an infinite integral of sums of quadratic functions of the outputs and the control inputs to the system. Using linear optimal control, this index supersedes all conventional performance criteria such as rise time, overshoot, damping ratio, etc. It becomes necessary then, to express control system performance in terms of the elements within the performance index. In order to select this performance index properly, it is important to be able to predict the closed-loop characteristics of the system in terms of conventional performance criteria. If linear optimal control can satisfy most conventional criteria, it will be a useful tool for linear system design. If it inherently provides additional advantages, then linear optimal control becomes an important tool in the design of linear control systems.

In this report, the relationships between conventional design criteria and optimal design criteria are investigated to determine whether or not optimal control can satisfy conventional design criteria; it can. In addition, other aspects of optimal control are investigated to determine if additional advantages exist by designing a system using linear optimal control. They do exist. Seldom, if ever, does a relatively new technique produce design advantages without limitations and disadvantages. Linear optimal control is restricted in its usage and does possess disadvantages if not properly used.

Linear optimal control is a general multivariable systhesis technique. Multi-controller, multi-output systems of high order can be conceptually designed very quickly using a digital computer and a unique control system will be specified for any one performance index. Using conventional techniques and criteria, a multi-controller design can be a tedious chore. Frequently,

the resulting system configuration will not be unique.

The use of linear optimal control techniques guarantees that the resulting closed-loop system will be stable. The complete right half of the complex frequency plane is eliminated as an area where closed-loop roots may exist. This feature of the technique can be important when the vehicle to be controlled is inherently unstable or flexible. The difficulty is that it may not always be possible to physically mechanize a system designed to stabilize an unstable vehicle or to minimize bending mode flexibility.

The closed-loop transient response of a linear optimal system tends to be smooth and well behaved. As the output is weighted heavily with respect to the control, the closed-loop response closely resembles the response of a Butterworth filter, whose transient response has little overshoot ($\zeta = .707$ for a second-order system) and whose frequency response is flat. Frequently, a linear optimal system has dynamic characteristics that an engineer strives for when using trial-and-error, conventional control system design procedures. Using optimal techniques, the controller motions can be qualitatively controlled. If a particular optimal design requires control input amplitudes larger than is desired, it is necessary only to increase the weighting of the control portion within the performance index, penalizing control motions more heavily. In this way, the control amplitudes may be reduced with, of course, an accompanying decrease in the speed of response of the closed-loop system.

The primary limitations of linear optimal control lie with the selection and interpretation of the performance index and the possible difficulty in physically mechanizing the resulting optimal control law. It appears easiest to select a performance index for the design of a completely automatic regulating system. For instance, it is not difficult to conceive of a performance index to satisfy many of the requirements of an automatic mid-air refueling system. In this application it is possible to identify dynamic variables that must be minimized, such as relative position errors between the two aircraft and the bending moments of the refueling aircraft. On the other hand, it is not known whether a quadratic performance index can be selected for the design of stability augmentation systems. Acceptable flying qualities are defined in terms of conventional dynamic criteria, such as short period natural frequency, damping ratio and lift curve slopes, and these quantities must be related to linear optimal design criteria before definite judgments can be made. However, because of the smoothness and generally well behaved dynamic characteristics of linear optimal systems, there is reason to believe that systems designed by linear optimal techniques will be judged acceptable for manual operations.

BACKGROUND

The solution to the linear optimal control problem probably evolved from the calculus of variations but its significance was not fully appreciated until R.E. Bellman and L.S. Pontryagin rigidly stated the conditions under which an optimum exists. The complete solution in the time domain was recently obtained by at least two prominent control system theorists, R.E. Kalman (Reference 1) and C.W. Merriam III (Reference 5). Kalman and Merriam have not only rigorously obtained mathematical proofs of the solution and

related theoretical aspects of the problem, but have been instrumental in developing digital computer programs for machine solution of large multi-controller problems. In the frequency domain, the original work by Wiener was extended by Mssrs. G. Newton, L. Gould and J. Kaiser. The problem with quadratic control constraint was solved completely in the scalar case by S.S.L. Chang (Reference 3).

Under Air Force Contract AF33(657)-7498, the Flight Research Department of the Cornell Aeronautical Laboratory investigated the application of linear optimal control techniques to several control system problems associated with aerospace vehicles using the digital computer program developed by T.S. Englar and R.E. Kalman (Reference 2).

It was found that the design technique has definite merit. Stable, well-behaved closed-loop systems can be conceptually specified using linear optimal control techniques. Large multivariable systems can be easily handled and many closed-loop optimal systems can be computed in a relatively short period of time. It was also discovered that, with practice, the control system designer could often qualitatively relate the parameters of the performance index to those dynamic characteristics known to yield an acceptable flight control system.

This report describes the results of an intensive study whose objective was to obtain relationships among the performance index parameters and the closed-loop optimal dynamics. The results show that the weighting parameters and the closed-loop poles are directly related. Both time and frequency domain approaches were used to minimize the integral performance index. The time domain approach used conventional calculus of variations techniques. The frequency domain approach uses Parseval's theorem in the manner advanced by S.S.L. Chang. Equivalence between the two methods can be demonstrated. The time domain approach uses the characteristic equation of the Euler-Lagrange and constraining equations to obtain a root square locus expression. The optimal control is shown to be governed by the matrix Riccati equation. The frequency domain approach shows that the conditions for optimality require the solution of a matrix Wiener-Hopf equation. A determinant can be extracted from the Wiener-Hopf equation that results in a root square locus expression for the closed-loop poles. The optimal control can be obtained by solving the Wiener-Hopf equation either by:

1. spectral factoring, or
2. a direct solution technique.

Many examples (both single-input and multi-input) are given in this report to demonstrate certain characteristics of linear optimal control. Examples of the root square locus and the equivalent Bode plots are numerous. Many of the examples use the equations of longitudinal aircraft motion to describe the object to be controlled.

The use of models to obtain a desirable closed-loop optimal system is also studied in this report. Specifically, the model can be included in the design objectives in two ways: the model can be included in the design as an input (as an uncontrollable part of the plant) or the model can be mathematically included in the performance index only. Examples to illustrate model procedures are given.

A significant start has been made on the problem of determining useful relationships between the feedback gains and the performance index. In the case of the single controller, the problem has been solved through the use of the root square locus expression as a design aid. For the single-input, single-output case, it is shown that a performance index can be formulated to yield specific feedback gains, closed-loop frequencies and damping, or specific steady state characteristics to a specified input.

The report describes the exact relationships that exist between the performance index parameters and the closed-loop optimal system poles. Because of this, one of the primary relationships between good aircraft flying qualities and optimal control characteristics has been established. However, more research is necessary to describe, in a usable, easily predictable form, the relationships between the optimal control law and the performance index.

This report places primary emphasis upon the relationships that exist among the parameters of the performance index and the resulting dynamic characteristics of the closed-loop optimal system. It is felt that a basic understanding of these relationships, and the optimal systems that they produce, is a prerequisite to their actual application. Optimal control can be another valuable addition to the design tools available to the flight control system designer. Some of the pertinent areas of application are emphasized in the examples included in this report. It is felt that this report will contribute to a better understanding of the technique, and accelerate its application to appropriate problems of control system design.

In Section 2 the optimal regulator of a second-order single-input, single-output system is obtained using the notation and basic techniques of R.E. Kalman, C.W. Merriam III, L.S. Pontryagin, S.S.L. Chang and a direct solution technique. The object of this section is to show that a unique optimal regulator is obtained regardless of the technique used. This section also shows that the techniques are basically the same, requiring the solution of a Riccati equation when formulated in the time domain and a Wiener-Hopf equation when formulated in the frequency domain.

In Section 3 of this report, the root square locus expression is developed. The integral is minimized by satisfying the Euler-Lagrange equations, whose characteristic determinant contains the closed-loop poles of the optimal system and adjoint. This characteristic determinant is manipulated into a root square locus expression. Performance indices containing control rates, output rates and models are considered, and the corresponding root square locus expressions are obtained.

Section 4 discusses some of the aspects of the single-input system, and shows that the problem is simply solved. The performance index can be related to the closed-loop dynamics and the optimal feedback gains. A performance index can be formulated to yield a predetermined feedback gain, including no feedback from a state variable, if desired. In general, however, negative values of q_{ii} in the performance index must be allowed.

Section 5 shows how optimal control techniques can be effectively used

to specify a control system design for the longitudinal short period control of a modern, high performance fighter aircraft.

The theory of the optimal control law for a system with a single control variable and a single output variable, using the frequency domain technique of S.S.L. Chang, is considered in Section 6. Several examples are given to illustrate the application of the method and to point out some difficult points which occur in the theory. The equivalence between Chang's method and the time domain approach is demonstrated.

The frequency domain approach of Section 6 is extended to the multivariable situation in Section 7. It is shown that the frequency domain relation of interest is a matrix equation of the Wiener-Hopf type. This matrix equation can then be solved using either of two methods:

1. spectral factorization, or
2. a direct method.

Examples are given to demonstrate the factorization approach and the direct method.

A subsection is included to show how one arrives at the matrix Wiener-Hopf equation when the basic description of the system is given in terms of transfer functions rather than a set of first-order differential equations. The section concludes with a theoretical development of the model-following technique and a method for synthesizing the feedback gains required by the optimal solution.

In Section 8, the use of Bode plots in linear optimal design is outlined in detail. A relatively complicated design problem, involving a jet fighter in a power approach, is used to illustrate the application of the concept. The section concludes with the outline of a frequency domain design procedure.

A more complex multivariable example of the use of the root square locus and the equivalent Bode plots is given in Section 9. The problems of dynamically matching a small jet to a proposed supersonic transport are illustrated using both the model-in-the-performance-index technique and the model-following concept.

This report is concerned with the solution of the problem involving a quadratic performance index. This index has shown to yield acceptable designs for many applications. One can always speculate on whether or not a better design would have been obtained if a different performance index had been used as the design criteria. Several excellent reports have been written describing the characteristics of systems designed using other performance indices (for instance, see References 16 and 17) and solutions to these problems using a suitable and realistic control constraint may eventually lead to a very useful set of design tools for the practicing engineer.

READER'S GUIDE

Most of the comments on the uses and abuses of linear optimal control are contained within the introduction and conclusions of this report. Those who are not mathematically minded, or those busy management people who decline to become too technically involved are urged to read only the Abstract,

Introduction and Conclusions.

Those who are more theoretically inclined would find Sections 3 and 7 most challenging. Section 3 contains most of the developments in the time domain and is the source for the derivation of the multi-output, multi-controller root square locus expression. Section 7 derives the frequency domain matrix Wiener-Hopf equation and describes a direct method for solving the Wiener-Hopf equation.

The engineer interested in optimal control, but either not familiar with matrix manipulations or who prefers relatively simple examples, will find that Sections 2, 4 and 6 will provide him with a fair understanding of the relationships between conventional and optimal design procedures.

Section 5 illustrates how the root square locus concept might be used for flight control system analysis and conceptual design, while Section 8 outlines the use of Bode plots in linear optimal design.

Finally, two multi-output, multi-controller examples are presented in Section 9 in connection with the use of models to obtain satisfactory and acceptable linear optimal control system designs.

Therefore, a reader may satisfy his curiosity about linear optimal control to any extent he desires. It is hoped that many will find the time to examine the contents of this report in detail, for it is believed by the authors that a powerful linear control system design technique will soon develop from linear optimal theory.

SECTION 2

SURVEY OF LINEAR OPTIMAL SOLUTION TECHNIQUES

2.1 INTRODUCTION

As a technical introduction to a study of linear optimal control, it was decided to review some of the solution techniques in use today. This review is accomplished by solving the same simple problem using several of these techniques. This section serves to illustrate that the techniques are quite similar, and all require the solution of a Riccati equation or its equivalent. Because the solution to the linear optimal control problem is unique, these similarities should come as no surprise. The main differences lie in the generality of the problem that can be solved, and these differences are briefly discussed at the end of the section. Although this section serves as background material, knowledge of the contents is not a prerequisite to understanding the technical developments of later sections.

We shall choose as a simple example the single-input, single-output second-order system completely describable by the transfer function

$$w(s) = \frac{y}{u}(s) = \frac{c}{s^2 + as + b} \tag{2-1}$$

It is desired to find the optimal control law that minimizes the integral

$$2V = \lim_{T\to\infty} \int_0^T (qy^2 + ru^2)\,dt \tag{2-2}$$

for any initial condition on the state vector.

2.2 THE METHOD ATTRIBUTABLE TO R. E. KALMAN (REFERENCE 1)

It is necessary to write the transfer function Equation 2-1 in first-order equation form,

$$\dot{x} = Fx + Gu \qquad y = Hx \tag{2-3}$$

Specifically, for this example there results

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2-4}$$

It has been proved (Reference 1) that the solution to this problem can be expressed as a feedback control law

$$u_0 = -Kx = -R^{-1}G'Px \tag{2-5}$$

where P is the steady state solution of the matrix Riccati equation

$$\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH \tag{2-6}$$

The matrix P is symmetrical and has two solutions. Kalman has shown that one solution will yield a realizable closed-loop system, guaranteed stability of the closed loop for the sufficient condition of non-negative definite Q and positive definite R matrices. This can be shown by considering the

integrand of the performance index to be a Lyapounov Function, but this proof will not be shown here.

The solution to the particular example of this section can be obtained by substituting the appropriate matrices into the Riccati equation (2-6) and solving for the steady state of the P matrix.

$$0 = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$
$$- \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ c \end{bmatrix} \begin{bmatrix} \frac{1}{r} \end{bmatrix} \begin{bmatrix} 0 & c \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} q \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (2\text{-}7)$$

This yields the three following scalar equations

$$\begin{aligned} 0 &= -2b p_{12} - \frac{p_{12}^2 c^2}{r} + q \\ 0 &= p_{11} - a p_{12} - b p_{22} - \frac{p_{12} p_{22} c^2}{r} \\ 0 &= 2 p_{12} - 2a p_{22} - \frac{p_{22}^2 c^2}{r} \end{aligned} \qquad (2\text{-}8)$$

Solving, there results for p_{12} and p_{22} :

$$p_{12} = -\frac{br}{c^2} + \frac{br}{c^2}\sqrt{1 + \frac{q}{r}\,\frac{c^2}{b^2}}$$

$$p_{22} = -\frac{ar}{c^2} + \frac{ar}{c^2}\sqrt{1 - \left(\frac{2b}{a^2} + \frac{2b}{a^2}\sqrt{1 + \frac{q}{r}\,\frac{c^2}{b^2}}\right)} \qquad (2\text{-}9)$$

The optimal feedback control law is given by

$$u_o = -Kx = -R^{-1}G'Px$$

$$= -\frac{1}{r}\begin{bmatrix} 0 & c \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{c}{r} p_{12} x_1 - \frac{c}{r} p_{22} x_2 \qquad (2\text{-}10)$$

The closed-loop optimal regulator is given by:

$$\dot{x} = (F - GK)x$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b - \frac{c^2}{r} p_{12} & -a - \frac{c^2}{r} p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2-11}$$

2.3 THE METHOD OF MERRIAM (REFERENCE 5)

The system is again written in the first-order form:

$$\dot{x} = Fx + Gu \qquad\qquad y = Hx$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2-12}$$

The performance index is

$$E(x(t), t) = \min_{u} \lim_{T \to \infty} \int_{\tau}^{T} \left[(q\, x_1(t)^2 + r\, u(t)^2 \right] dt$$

where

$$\tau \le t \le T$$

Richard Bellman has proven that the relationship that minimizes E is given by

$$\min_{u(t)} \left[q x_1(t)^2 + r u(t)^2 + \frac{dE}{dt} \right] = 0 \tag{2-13}$$

where

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} \dot{x} \tag{2-14}$$

Substituting 2-12 and 2-14 into 2-13, and realizing that as T approaches infinity E(t) approaches a constant value and $\partial E / \partial t = 0$, the result is

$$\min_u \left\{ q x_1^2 + r u^2 + \frac{\partial E}{\partial x_1} (+x_2) + \frac{\partial E}{\partial x_2} (-b x_1 - a x_2 + cu) \right\} = 0 \tag{2-15}$$

Differentiating with respect to u yields the control law that minimizes Equation 2-15, and therefore the performance index.

$$2ru + c \frac{\partial E}{\partial x_2} = 0$$

$$u = -\frac{c}{2r} \frac{\partial E}{\partial x_2} \tag{2-16}$$

Merriam now assumes the following form for E:

$$E = k - 2 \sum_{m=1}^{n} k_m(t)\, x_m(t) + \sum_{m=1}^{n} \sum_{p=1}^{n} k_{mp}(t)\, x_m(t)\, x_p(t)$$

$$= k - 2(k_1 x_1 + k_2 x_2) + k_{11} x_1^2 + k_{12} x_1 x_2 + k_{21} x_2 x_1 + k_{22} x_2^2 \tag{2-17}$$

where the k's are constants resulting from the requirement that $T \rightarrow \infty$ in the performance index.

Then, assuming that $k_{12} = k_{21}$

$$\frac{\partial E}{\partial x_1} = 2(-k_1 + k_{11} x_1 + k_{12} x_2)$$

$$\frac{\partial E}{\partial x_2} = 2(-k_2 + k_{12} x_1 + k_{22} x_2) \tag{2-18}$$

Substituting Equation 2-18 into 2-16 yields

$$u_0 = \frac{c}{r}(k_2 - k_{12} x_1 - k_{22} x_2) \tag{2-19}$$

Now substituting 2-19 into 2-15 yields an expression from which the constants k can be determined.

$$q x_1^2 + \frac{c^2}{r}(k_2 - k_{12} x_1 - k_{22} x_2)^2 + 2 x_2(-k_1 + k_{11} x_1 + k_{12} x_2)$$

$$+ 2(-k_2 + k_{12} x_1 + k_{22} x_2)\left[-b x_1 - a x_2 + \frac{c^2}{r}(k_2 - k_{12} x_1 - k_{22} x_2)\right] = 0 \tag{2-20}$$

Expanding and equating terms of the same powers of x to zero yields the following set of quadratic equations:

Powers in x^0 $\quad \frac{c^2}{r} k_2^2 - 2\frac{c^2}{r} k_2^2 = 0$ a)

x_1 $\quad 2k_2\left(b + k_{12}\frac{c^2}{r}\right) = 0$ b)

x_1^2 $\quad q - 2k_{12} b - k_{12}^2 \frac{c^2}{r} = 0$ c) (2-21)

$x_1 x_2$ $\quad 2k_{11} - 2bk_{22} - 2a k_{12} - 2k_{12} k_{22}\frac{c^2}{r} = 0$ d)

x_2 $\quad -2k_1 + 2a k_2 + 2k_2 k_{22}\frac{c^2}{r} = 0$ e)

x_2^2 $\quad 2k_{12} - 2k_{22} a - \frac{c^2}{r} k_{22}^2 = 0$ f)

It is clear that Equations 2-21c, d and f are the same as Equations 2 - 8. It can also be seen from Equation 2-21a and e that k_1 and k_2 are zero for this example. Therefore,

$$k_{ij} = p_{ij}$$

k_{ij} of Equation 2-21

p_{ij} of Equation 2-8

and Equation 2-16 may be written the same as Equation 2-10, leading to the identical closed-loop optimal system.

2.4 PONTRYAGIN'S TECHNIQUE (REFERENCE 4)

Pontryagin also uses the first-order form to define a linear system:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -bx_1 - ax_2 + cu$$

The performance index is defined as

$$x_0 = \lim_{t_1 \to \infty} \int_{t_0}^{t_1} (qx_1^2 + ru^2)dt$$

To the original system a new vector is added, and the new system is defined as follows:

$$\begin{aligned} f^0 &= \dot{x}_0 = qx_1^2 + ru^2 \\ f^1 &= \dot{x}_1 = x_2 \\ f^2 &= \dot{x}_2 = -bx_1 - ax_2 + cu \end{aligned} \tag{2-22}$$

A Hamiltonian function $\mathcal{H} = \sum_{\alpha=0}^{n} \psi_\alpha f^\alpha$ is generated, which for this particular example becomes:

$$\mathcal{H} = \psi_0 (qx_1^2 + ru^2) + \psi_1 x_2 + \psi_2 (-bx_1 - ax_2 + cu) \tag{2-23}$$

A second function $M = \sup_u \mathcal{H}$ is defined. From theorem 1 (Reference 4, page 19), $M(\psi, x) = 0$ and since the system is linear and continuous,

$$M(\psi, x) = \sup_u \mathcal{H}(\psi, x) = \frac{\partial \mathcal{H}}{\partial u} = 0$$

Substituting, there results

$$\frac{\partial \mathcal{H}}{\partial u} = 2\psi_0 ru_0 + \psi_2 c = 0$$

from which the optimal control law is obtained

$$u_0 = -\frac{c}{2r}\frac{\psi_2}{\psi_0} \tag{2-24}$$

Substituting 2-24 into 2-23 gives the result

$$0 = \left[q x_1^2 + r \left(\frac{-c}{2r} \; \frac{\psi_2}{\psi_0} \right)^2 \right] + \psi_1 x_2 + \psi_2 \left[-b x_1 - a x_2 + c \left(\frac{-c}{2r} \; \frac{\psi_2}{\psi_0} \right) \right] \qquad (2\text{-}25)$$

The Hamiltonian is such to satisfy the Hamiltonian system of partial differential equations

$$\begin{aligned} \dot{\psi}_0 &= 0 \\ \dot{\psi}_i &= -\frac{\partial \mathcal{H}}{\partial x_i} \qquad i = 1, 2, \ldots n \\ \dot{x}_i &= +\frac{\partial \mathcal{H}}{\partial \psi_i} \qquad i = 1, 2, \ldots n \end{aligned} \qquad (2\text{-}26)$$

Therefore, ψ_0 is a constant and can be set equal to 1. Performing the operations of Equation 2-26, the result is

$$\begin{aligned} \dot{\psi}_1 &= -2q x_1 + b \psi_2 && \text{a)} \\ \dot{\psi}_2 &= -\psi_1 + a \psi_2 && \text{b)} \\ \dot{x}_1 &= x_2 && \text{c)} \\ \dot{x}_2 &= -b x_1 - a x_2 - \frac{c^2 \psi_2}{2r} && \text{d)} \end{aligned} \qquad (2\text{-}27)$$

Equations 2-27 define the optimal system and they must be solved to obtain the expression for the synthesis of the optimal system. It can be shown that a solution will be obtained by assuming

$$\begin{aligned} \psi_1 &= \phi_{11} x_1 + \phi_{12} x_2 \\ \psi_2 &= \phi_{12} x_1 + \phi_{22} x_2 \end{aligned}$$

where the ϕ_{ij} are constant for this problem.

$$\begin{aligned} \dot{\psi}_1 &= \phi_{11} \dot{x}_1 + \phi_{12} \dot{x}_2 \\ \dot{\psi}_2 &= \phi_{12} \dot{x}_1 + \phi_{22} \dot{x}_2 \end{aligned} \qquad (2\text{-}28)$$

Substituting 2-28 and 2-27c and d into 2-27a and b, and grouping terms in x_1 and x_2 yields

$$x_1\left(\frac{-c^2}{2r}\,\phi_{12}^{\,2} - 2b\phi_{12} + 2q\right) + x_2\left(\frac{-c^2}{2r}\,\phi_{12}\phi_{22} - a\phi_{12} - b\phi_{22} + \phi_{11}\right) = 0$$

$$x_1\left(\frac{-c^2}{2r}\,\phi_{12}\phi_{22} - a\phi_{12} - b\phi_{22} + \phi_{11}\right) + x_2\left(\frac{-c^2}{2r}\,\phi_{22}^{\,2} - 2a\phi_{22} + 2\phi_{12}\right) = 0 \qquad (2\text{-}29)$$

Equation 2-29 is satisfied if the following three equations are satisfied.

$$\frac{-c^2}{2r}\,\phi_{12}^{\,2} - 2b\phi_{12} + 2q = 0$$

$$\frac{-c^2}{2r}\,\phi_{12}\phi_{22} - a\phi_{12} - b\phi_{22} + \phi_{11} = 0 \qquad (2\text{-}30)$$

$$\frac{-c^2}{2r}\,\phi_{22}^{\,2} - 2a\phi_{22} + 2\phi_{12} = 0$$

But Equations 2-30 are identical to the Riccati equations 2-8 when ϕ_{ij} is set equal to $2\rho_{ij}$ again demonstrating that the optimal system requires a solution of the same set of equations and, of course, yields identical results.

2.5 THE METHOD OF CHANG (REFERENCE 3)

Chang's method was designed for deterministic and statistical inputs, and is not directly applicable to the regulator problem. However, using the theory extension described in Section 6.3, the optimal system can be obtained.

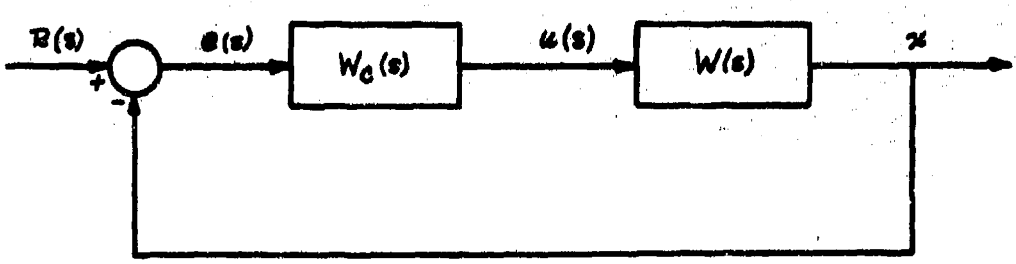


Figure 1. Single Output Block Diagram

The block diagram of Figure 1 where

- W(s) = fixed system elements (transfer function)
- u(s) = control (input to the fixed elements)
- R(s) = closed-loop system input
- e(s) = error signal
- $W_c(s)$ = compensating network (to be designed once the optimal u is specified)

leads to the Wiener-Hopf equation:

$$\left[k^2 + W(s)W(-s)\right]u_0 - W(-s)R(s) = g(s) \tag{2-31}$$

when the performance index is

$$2V = \int_0^\infty (e^2 + k^2 u^2)dt \tag{2-32}$$

In Equation 2-31, $g(s)$ represents a rational polynomial that can have no poles in the left-half plane while the optimal control, u_0, can have no poles in the right-half plane.

The solution to Equation 2-31 is

$$u_0 = \frac{1}{Y(s)}\left[\frac{W(-s)R(s)}{Y(-s)}\right]_+ \tag{2-33}$$

where the symbol $[\quad]_+$ means that one expands $\dfrac{W(-s)R(s)}{Y(-s)}$ in partial fractions and retains only those terms with left-half plane poles.

The term $Y = \left\{k^2 + W(s)W(-s)\right\}^+$

is found by factoring $k^2 + W(s)W(-s)$

into a product with one component having all its poles and zeros in the left-half plane while the other has right-half plane poles and zero. One then picks $\left\{k^2 + W(s)W(-s)\right\}^+$ to be the left-half plane factor.

For this example, the equivalent regulator formulation is

$$W(s) = \frac{c}{s^2 + as + b}$$

$$R(s) = -\frac{[(s+a)x_1(0) + x_2(0)]}{s^2 + as + b}$$

where a, b, and c have the same meaning as the previous examples. $x_1(0)$ and $x_2(0)$ are initial conditions. Applying Equation 2-31, one finds

$$\left[k^2 + \frac{c^2}{(s^2 - as + b)(s^2 + as + b)}\right] u_0 + \frac{c}{s^2 - as + b}\left[\frac{(s+a)x_1(0) + x_2(0)}{s^2 + as + b}\right] \tag{2-34}$$

By direct substitution into Equation 2-33, the optimal control is found to be

$$u_0 = \frac{1}{\left\{k^2 + \dfrac{c^2}{(s^2 - as + b)(s^2 + as + b)}\right\}^+}\left[\frac{c\left[(s+a)x_1(0) + x_2(0)\right]}{(s^2 - as + b)(s^2 + as + b)\left\{k^2 + \dfrac{c^2}{(s^2 - as + b)(s^2 + as + b)}\right\}^-}\right]_+ \tag{2-35}$$

which simplifies to

$$u_0 = \frac{s^2+as+b}{k^2(s^2+\alpha s+B)} \left[\frac{c\left[(s+a)x_1(0)+x_2(0)\right]}{(s^2-\alpha s+B)(s^2+\alpha s+B)} \right]_+ \qquad (2\text{-}36)$$

where

$$\left[s^4-(a^2-2b)s^2+b^2+\frac{a^2}{k^2} \right] = (s^2-\alpha s+B)(s^2+\alpha s+B) \qquad (2\text{-}37)$$

In Equation 2-36, only the roots of the equation $s^2 + as + b = 0$ contribute to the partial fraction expansion.

Solving for B and α, in terms of a and b, one finds

$$\omega_0^2 = B = \sqrt{b^2+\frac{a^2}{k^2}}$$
$$\alpha = \sqrt{a^2-2b+2\sqrt{b^2+\frac{a^2}{k^2}}} \qquad (2\text{-}38)$$

The closed-loop natural frequency is given by Equation 2-38 and agrees with the results obtained by the other methods.

State vectors do not appear in this frequency domain formulation of the problem and one cannot use the equation

$$u_0 = -Kx$$

to synthesize the feedback configuration. However, one can choose to work with the block diagram of Figure 2 and the performance index

$$2V = \int_0^\infty (e^2+k^2u^2)dt$$

where $e = x - c$ is the system error. The feedback found would be identical to that found using the other solution techniques.

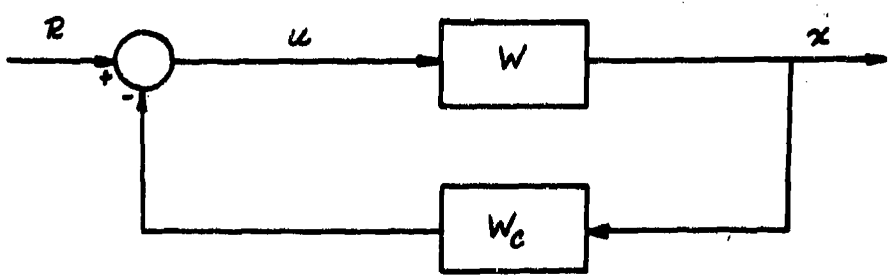


Figure 2. Alternate Block Diagram

2.6 DIRECT SOLUTION

It can be shown that there is really no need, for relatively simple single-input, single-output systems as used in this example, to resort to the step-by-step procedures of optimal control system synthesis outlined in using the techniques of Kalman, Pontryagin, Merriam or Chang in determining the optimal feedback control law. The feedback gains can be obtained directly from the expressions for the optimal closed-loop regulator and its adjoint (i.e., its right-half plane mirror image). The optimal feedback control law is shown by Kalman to be of the form

$$u_0 = -Kx$$

and the closed-loop optimal system is

$$\dot{x} = (F-GK)x$$

whose characteristic equation is

$$\left| Is-(F-GK) \right| = 0 \tag{2-39}$$

The spectral factored product of the characteristic equation of the optimal system and its adjoint is given by

$$\left| Is-(F-GK) \right| \left| -Is-(F-GK)' \right| = 0 \tag{2-40}$$

It will be shown in Section 3, specifically Equation 3-20, that the characteristic equation of the optimal system and its adjoint is given by:

$$\begin{vmatrix} Is-F & GR^{-1}G' \\ -H'QH & -Is-F' \end{vmatrix} = 0 \tag{2-41}$$

Equating Equations 2-40 and 2-41 will yield expressions relating the feedback gains K with F, G, H, Q and R of the optimal system. There results

$$\left| Is-(F-GK) \right| \left| -Is-(F-GK)' \right| = \begin{vmatrix} Is-F & GR^{-1}G' \\ -H'QH & -Is-F' \end{vmatrix} \tag{2-42}$$

Substituting the parameters of the particular example that is being used,

$$\begin{vmatrix} s & -1 \\ b+ck_1 & s+a+ck_2 \end{vmatrix} \begin{vmatrix} -s & b+ck_1 \\ -1 & -s+a+ck_2 \end{vmatrix} = \begin{vmatrix} s & -1 & 0 & 0 \\ b & s+a & 0 & c^2/r \\ -q & 0 & -s & b \\ 0 & 0 & -1 & -s+a \end{vmatrix}$$

or

$$\left[s^2+s(a+ck_2)+(b+ck_1)\right]\left[s^2-s(a+ck_2)+(b+ck_1)\right] = (s^2+as+b)(s^2-as+b)+\frac{c^2q}{r}$$

Therefore:

$$s^4 - s^2\left[(a+ck_2)^2 - 2(b+ck_1)\right] + (b+ck_1)^2 = s^4 - s^2(a^2-2b) + b^2 + \frac{c^2q}{r} \qquad (2\text{-}43)$$

Equating powers of s yields:

$$k_2^2 + \frac{2ak_2}{c} - \frac{2}{c}k_1 = 0$$

$$k_1^2 + \frac{2bk_1}{c} - \frac{q}{r} = 0$$

$$k_1 = -\frac{b}{c} \pm \sqrt{\frac{b^2}{c^2} + \frac{q}{r}} = -\frac{b}{a} \pm \frac{b}{c}\sqrt{1 + \frac{q}{r}\frac{c^2}{b^2}}$$

$$k_2 = -\frac{a}{c} \pm \sqrt{\frac{a^2}{c^2} + \frac{2}{c}k_1} \qquad (2\text{-}44)$$

These feedback gains again lead to the same optimal systems as obtained by the other techniques.

The advantages of this direct solution are clear. The feedback gains are expressed directly as function of q and r. The right-hand side of Equation 2-42 is the expression from which the root square locus is derived. A root square locus can be performed beforehand, yielding the values of the closed-loop left-hand plane roots. A polynomial can be formed by these roots and equated directly to the characteristic equation of the optimal system, Equation 2-39. The feedback gains would then be obtained as linear functions of the coefficients of the polynomial formed from the root square locus plot.

A numerical example of this technique is given in Section 5 where it is also shown that the q's and r's can be chosen to yield feedback gains from selected parameters of the system. It should be cautioned, however, that this technique has been found to work only for single-input systems. If a multivariable optimal system must be designed, the Riccati equation must be solved or the technique of Section 7 must be used.

The similarities between these techniques of solution for linear systems are apparent. They all lead to a set of quadratic equations, the matrix Riccati equation. The method of proof of the existence of the optimum varies somewhat between the different methods of solution, but the greatest difference among the techniques is the notational language used. The similarities can be summarized briefly in a few paragraphs.

Bellman, of course, is famous for his Principle of Optimality and dynamic programming. He has demonstrated that the optimal solution using the performance criterion

$$V = \int_{t_0}^{t_1} (Qx^2 + Ru^2)dt \qquad (2\text{-}45)$$

is obtained from the equation

$$\min_u \left\{ Qx^2 + Ru^2 + \frac{\partial V_{opt}}{\partial t} + \frac{\partial V_{opt}}{\partial x} \cdot \dot{x} \right\} = 0 \tag{2-46}$$

where

$$V_{opt}\ (t, x) = \min_u \int_{t_0}^{t_1} (Qx^2 + Ru^2)\, dt$$

and Merriam obtained a synthesis of the optimal system in closed form by assuming a form for V_{opt} :

$$V_{opt}\ (t, x) = K_0(t) + \sum_i^n K_i(t)\, x_i(t) + \sum_i^n \sum_j^n K_{ij}(t)\, x_i(t)\, x_j(t) \tag{2-47}$$

Kalman demonstrated that Equation 2-46 was essentially the Hamilton-Jacobi partial differential equation

$$\frac{\partial V_{opt}}{\partial t} + \mathcal{H}_{opt}(x, p, t) = 0 \tag{2-48}$$

where

$$\mathcal{H}_{opt}(x, p, t) = \min_u \left\{ Qx^2 + Ru^2 + p \cdot \dot{x} \right\} \tag{2-49}$$

where Kalman's

$$p = \frac{\partial V_{opt}}{\partial x} \tag{2-50}$$

Kalman also demonstrated that the solution to the Hamilton-Jacobi equation is equivalent to a solution of the canonical equations

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}_{opt}}{\partial p} \\ \dot{p} &= -\frac{\partial \mathcal{H}_{opt}}{\partial x} \end{aligned} \tag{2-51}$$

If the space X x U is unbounded, the canonical equations can be reduced to the Riccati quadratic first-order differential equation

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH \tag{2-52}$$

by substituting p = Px into the canonical equations. The matrices F, G and H are the plant matrices as defined by Kalman.

Pontryagin has proven the existence of the function $\mathcal{H} - \psi \cdot \dot{x}$, which when maximized with respect to the control u, can lead to the optimum control system. It is clear that Pontryagin's adjoint variable, ψ , Kalman's co-state, p, and Bellman's gradient vector, $\partial V_{opt} / \partial x$, are related by constants for the linear system.

For the simple example illustrated in this section, any of these solution techniques can be used with about equal ease. The major differences in the methods lie with the types of problems that can be solved.

The maximum principle appears applicable to almost every type of dynamic system, linear or nonlinear, and the widest variety of performance criterion. It is not a solution technique, however. The maximum principle states the conditions under which an optimum exists, but it is up to the ingenuity of the design engineer to find a solution to the Hamiltonian system of equations that maximizes the Hamiltonian function.

It appears that the parameter expansion method of Merriam and the method attributable to Kalman are capable of solving the same variety of problems. Time-varying systems, having finite or infinite performance index integrals, are handled by either method and there appears to be no basic limitation to the order of the system or the number of control inputs.

The frequency domain solution method requires a performance index in which the upper limit is infinity and a plant that has a Laplace transform description. These requirements admit the existence of a transport lag, or time delay, which do not invalidate the solution technique. The frequency domain approach to linear optimal control shows how a time delay is to be treated. This is not apparent when using the time domain approach of Merriam or Kalman.

The direct solution technique is the simplest and the easiest to perform, particularly if a root square locus plot is used to spectral factor the poles of the optimal system and its adjoint. At present, this technique applies only to single-input systems, but it should be extendable to multi-input systems.

SECTION 3

THE ROOT SQUARE LOCUS

3.1 THE GENERAL PROBLEM

S.S.L. Chang (Reference 3) has shown that there can be associated with optimal systems involving a quadratic performance index, a root square locus plot involving the poles of the optimal system and the adjoint (right-half plane image) system. Dr. Chang considered only single-control, single-output systems, however, and it is of definite interest to expand this concept. A digital program now exists to obtain the optimum of large, multivariable systems, but there is no quantitative method for predicting the closed-loop roots of the optimum system. A multivariable root square locus expression will help the control system designer relate the parameters of the performance index to the dynamics of the closed-loop optimal system. It is desired to obtain a matrix multivariable expression for the poles of the closed-loop optimal system. It is also of interest to obtain expressions for the root square locus of quadratic performance index forms containing the derivatives of the control and the output variables.

The use of an integral whose upper limit approaches infinity, and whose integrand is a quadratic function of the state and the control variables of a linear system, to express control system requirements can be formulated as a standard problem in the theory of the Calculus of Variations. The exact problem is treated in almost any standard text (see, for instance, C. Fox, "Introduction to the Calculus of Variations", Oxford, 1950, Sec. 4.8, p. 94).

The general variational problem treated is to find the extremum of the integral

$$V = \int_{t_1}^{t_2} \mathcal{N}(x, \dot{x}, u, \dot{u}, t)\, dt \tag{3-1}$$

subject to the differential constraint

$$\mathcal{M}(x, \dot{x}, u, \dot{u}) = 0 \tag{3-2}$$

By taking the appropriate variations (see Fox), the two classical Euler-Lagrange partial differential equations are obtained: (two equations because there are two dependent variables)

$$\frac{\partial \mathcal{N}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{N}}{\partial \dot{x}}\right) - \left[\lambda \frac{\partial \mathcal{M}}{\partial x} - \frac{d}{dt}\left(\lambda \frac{\partial \mathcal{M}}{\partial \dot{x}}\right)\right] = 0 \tag{3-3}$$

(λ an arbitrary function of time that is to be determined)

$$\frac{\partial \mathcal{N}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{N}}{\partial \dot{u}}\right) - \left[\lambda \frac{\partial \mathcal{M}}{\partial u} - \frac{d}{dt}\left(\lambda \frac{\partial \mathcal{M}}{\partial \dot{u}}\right)\right] = 0 \tag{3-4}$$

These Euler-Lagrange equations can be obtained another way. Define a function

$$\mathcal{L} = \eta + \lambda' \cdot m \tag{3-5}$$

The Euler-Lagrange equations are then simply:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \tag{3-6}$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0 \tag{3-7}$$

The phase of linear optimal control of immediate interest falls within the general form of Equations 3-1 and 3-2 with, however, several important limitations.

1. The constraining equation is linear.
2. The integrand contains quadratic forms only.
3. The limits of the integral are usually taken between zero and infinity.

With these alterations, the problem can be defined as follows: Determine the control u that minimizes the integral

$$2V = \int_0^{\infty} \left(x'Qx + u'Ru + \dot{x}'S\dot{x} + \dot{u}'T\dot{u} + \begin{matrix}\text{scalar products}\\ \text{of } x, \dot{x}, u, \dot{u}\end{matrix}\right) dt \tag{3-8}$$

subject to the constraint

$$-\dot{x} + Fx + Gu = 0 \qquad y = Hx \tag{3-9}$$

where Q and S are non-negative definite symmetric matrices (the non-negative requirement guarantees stability, a sufficient but not a necessary condition). R and T are positive definite.

F is an n x n system matrix

G is an n x p input matrix describing the effect of an input on the system.

The solutions to the problem stated above require at least piecewise existence of the second time derivative of the dependent variables x and u .

3.2 THE ROOT SQUARE LOCUS DERIVATION

The linear optimal problem usually formulated is a simplification of the more general equations (3-8 and 3-9). The integral to be minimized is:

$$2V = \int_0^{\infty} \eta(x,u)\,dt = \int_0^{\infty} (y'Qy + u'Ru)\,dt = \int_0^{\infty} (x'H'QHx + u'Ru)\,dt \tag{3-10}$$

The constraining equation is, as usual,

$$m(\dot{x}, x, u) = -\dot{x} + Fx + Gu = 0 \tag{3-11}$$

The Lagrangian is formed

$$\mathcal{L} = \frac{1}{2} n + \lambda' \cdot m = \frac{1}{2}(x'H'QHx + u'Ru) + \lambda' \cdot (-\dot{x} + Fx + Gu)$$

Obtaining the gradients as indicated by the Euler equations (3-6 and 3-7),

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}\left[(H'QH)'x + H'QHx\right] + F'\lambda = H'QHx + F'\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\lambda$$

$$\frac{\partial \mathcal{L}}{\partial u} = \frac{1}{2}(R'u + Ru) + G'\lambda = Ru + G'\lambda$$

The Euler equations reduce to

$$\dot{\lambda} + H'QHx + F'\lambda = 0 \tag{3-12}$$

$$Ru + G'\lambda = 0 \tag{3-13}$$

Solving Equation 3-13 for u yields the control (u_0) that minimizes the integral, Equation 3-10:

$$u_0 = -R^{-1}G'\lambda \tag{3-14}$$

There are three equations then that define the optimal system; the two Euler equations and the restraining equation. These three equations are, in a partitioned matrix form:

$$\begin{bmatrix} \dot{x} \\ 0 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} F & G & 0 \\ 0 & R & G' \\ -H'QH & 0 & -F' \end{bmatrix} \begin{bmatrix} x \\ u \\ \lambda \end{bmatrix} \begin{matrix} \text{(a)} \\ \text{(b)} \\ \text{(c)} \end{matrix} \tag{3-15}$$

It has been shown by Kalman and others that the control law is governed by a matrix equation called the Riccati equation. To obtain this matrix Riccati equation, substitute Equation 3-14 into 3-15a and consider the resulting equation along with 3-15c:

$$\begin{aligned} \dot{x} - Fx + GR^{-1}G'\lambda &= 0 \quad \text{(a)} \\ H'QHx + \dot{\lambda} + F'\lambda &= 0 \quad \text{(b)} \end{aligned} \tag{3-16}$$

Then let $\lambda = Px$, $\dot{\lambda} = P\dot{x} + \dot{P}x$. Substituting for $\dot{\lambda}$ into Equation 3-16b and multiplying Equation 3-16a by P yields:

$$\begin{aligned} P\dot{x} - PFx + PGR^{-1}G'Px &= 0 \quad \text{(a)} \\ H'QHx + P\dot{x} + \dot{P}x + F'Px &= 0 \quad \text{(b)} \end{aligned} \tag{3-17}$$

P is a symmetric matrix which is a function of time

Solving for $P\dot{x}$ in Equation 3-17a and substituting into Equation 3-17b yields

$$-\dot{P}x = PFx + F'Px - PGR^{-1}G'Px + H'QHx$$

or

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH \qquad (3\text{-}18)$$

which is the matrix Riccati equation.

Kalman (Reference 1) has shown that $x(0)'P(\infty)x(0)$ is the optimum value of the performance index, whose value approaches a constant as the upper limit of the performance index approaches infinity. Because this report only treats the performance index whose upper limit is infinite, the steady state solution of the Riccati equation is required. Setting the left-hand side of Equation 3-18 to zero yields the solution for $P(\infty)$, and therefore the value of $\lambda = P(\infty)x(t)$ for the optimal feedback control law of Equation 3-14.

The Riccati equation is a matrix quadratic equation in P. It has, therefore, two solutions and it is found that one solution yields a stable closed-loop optimal system, and the other produces an unstable adjoint, or image solution. Because it can be shown that the performance index V is a Lypanounov function (Reference 1), the optimal closed-loop system for this case is stable. The Riccati equation therefore yields an optimal and stable closed-loop solution.

To demonstrate the character of the closed-loop roots of the optimal system and its adjoint, take the Laplace transform of Equation 3-16a and the negative of Equation 3-16b

$$\left[\begin{array}{c|c} Is-F & GR^{-1}G' \\ \hline -H'QH & -Is-F' \end{array}\right] \begin{bmatrix} x(s) \\ \lambda(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ -\lambda(0) \end{bmatrix} \qquad (3\text{-}19)$$

The characteristic equation of the closed-loop set of Equations 3-19 is given by the determinant expression

$$\Delta(-s)\Delta(s) = \begin{vmatrix} Is-F & GR^{-1}G' \\ -H'QH & -Is-F' \end{vmatrix} = 0 \qquad (3\text{-}20)$$

This determinant contains the closed-loop poles of the complete optimal solution; therefore it must have a stable left-half plane set of poles and an unstable set of closed-loop poles. Letov (Reference 15) has proven that if $\mu_i(s)$ is a root of Equation 3-20, $\mu_i(-s)$ must also be a root. Assume that the terms H'QH and $GR^{-1}G'$ are zero, that is, there is no performance index associated with the problem, and the system is open loop. Under these conditions, the adjoint system is merely the adjoint of the plant. Equation 3-20 becomes

$$\Delta(-s)\Delta(s) = \begin{vmatrix} Is-F & 0 \\ 0 & -Is-F' \end{vmatrix} = \left|Is-F\right| \left|-Is-F'\right| = 0 \qquad (3\text{-}21)$$

Equation 3-21 then expresses a system whose roots are images of each other reflected about the $j\omega$ axis of a complex frequency s-plane, where

$s = \sigma + j\omega$. Notice that if the original system $\dot{x} = Fx + Gu$ is unstable, its adjoint $\dot{x} = -F'x$ is stable and vice versa. Also, if the original plant contained both a stable and an unstable part, the adjoint would have both a stable and an unstable part, with the unstable part of the plant now the stable part of the adjoint. Equation 3-21 contains two complete sets of poles. In the left-half plane are the stable part of the plant <u>and</u> the stable part of the adjoint. The right-half plane contains the unstable plant poles and the reflected stable plant poles. Now add a performance index to the problem, i.e., assume that $H'QH$ and $GR^{-1}G'$ terms in Equation 3-20 are finite, with the $GR^{-1}G'$ term much smaller than the $H'QH$ term. The roots of the determinant of Equation 3-20 change slightly from their open-loop values of Equation 3-21. The significant observation is that those roots of Equation 3-20 that start out in the left-half plane always remain in the left-half plane (for Q non-negative definite and R positive definite) and those roots in the right-half plane remain in the right-half plane. The roots in the left-half plane are the poles of the optimal realizable system. <u>It will be shown that the roots of Equation 3-20, and therefore the poles of the optimal system (and its adjoint) move, as a very definite function of Q and R, in a manner describable by a root square locus.</u>

Before an expression for the multivariable root square locus is developed, it is important to notice that if the system matrix F is of order n, the determinant of Equation 3-19, which defines the closed-loop system and its adjoint, contains exactly 2n closed-loop roots. The Riccati equation has 2n solutions as well; n solutions which define a stable closed-loop system and n which define an unstable system. The realizable closed-loop system will therefore have n stable poles, exactly the same number as the open-loop system.

The determinant of Equation 3-19 defines the roots of the closed-loop optimal system and its adjoint, and it can certainly be used to find these poles, but a more convenient and useful expression can be developed to conform with S.S.L. Chang's single-input, single-output expression of Reference 3.

The variational equations obtained after taking a Laplace transform of Equations 3-15 and rearranging, are:

$$\begin{bmatrix} -Is-F' & -H'QH & 0 \\ 0 & Is-F & -G \\ G' & 0 & R \end{bmatrix} \begin{bmatrix} \lambda(s) \\ x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} -\lambda(0) \\ x(0) \\ 0 \end{bmatrix} \qquad (3-22)$$

where the roots of the optimal system and its adjoint are given by

$$\Delta(s)\,\Delta(-s) = \begin{vmatrix} -Is-F' & -H'QH & 0 \\ 0 & Is-F & -G \\ G' & 0 & R \end{vmatrix} = 0 \qquad (3-23)$$

This determinant can be conveniently expanded by using the <u>Generalized Algorithm of Gauss</u> (see, for instance, F.R. Gantmacher, "Matrix Theory", Chelsea Publishing, 1960). The resulting expression is

$$\Delta(s)\,\Delta(-s) = \left|-Is - F'\right|\left|Is - F\right|\left|R + G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G\right| = 0 \qquad (3\text{-}24)$$

which holds if $[Is - F]$, $[-Is - F']$ are square and for values of s such that $|Is - F| \neq 0$, $|-Is - F'| \neq 0$. The determinants $|Is - F|$ and $|-Is - F'|$ are the determinants that define the characteristic equations of the open-loop system and its adjoint. By definition, they are the determinants of square matrices and are equal to zero only at the values of s equal to the open-loop poles of the system.

Defining $\quad |Is - F| = D \qquad |-Is - F'| = \bar{D}$

Equation 3-24 can be written

$$\Delta(s)\,\Delta(-s) = D\bar{D}\left|R + G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G\right| = 0$$

However, since $D\bar{D} = 0$ defines the poles of the open-loop plant and its adjoint, they are not part of the closed-loop optimal system and adjoint. The scalar expression that defines the locus of poles of the closed-loop system and its adjoint is given by

$$\left|R + G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G\right| = 0$$

or

$$\left|I + R^{-1}G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G\right| = 0 \qquad (3\text{-}25)$$

Equation 3-25 defines a root square locus. The form is conventional and all of the root locus techniques now in use can be employed to solve Equation 3-25.

It is shown in Appendix I that Equation 3-25 can be written in the form

$$\left|I + R^{-1}\left[\frac{Y}{U}(-s)\right]'Q\left[\frac{Y}{U}(s)\right]\right| = 0 \qquad (3\text{-}26)$$

where, by definition,

$$H[Is-F]^{-1}G = \left[\frac{Y}{U}(s)\right] = \begin{array}{cc} & \text{SAME OUTPUT VARIABLE} \longrightarrow \\ \begin{array}{c}\text{SAME}\\\text{INPUT}\\\text{VARIABLE}\\ \downarrow\end{array} & \begin{bmatrix} \frac{y_1}{u_1}(s) & \frac{y_1}{u_2}(s) & \cdots & \frac{y_1}{u_p}(s) \\ \frac{y_2}{u_1}(s) & \frac{y_2}{u_2}(s) & \cdots & \frac{y_2}{u_p}(s) \\ \vdots & \vdots & & \vdots \\ \frac{y_n}{u_1}(s) & \frac{y_n}{u_2}(s) & \cdots & \frac{y_n}{u_p}(s) \end{bmatrix} \end{array}$$

(3-27)

is a matrix of transfer or weighting functions of the n outputs to the p inputs to the open-loop system. It has been found convenient to use the symbol $W(s) = \left[\frac{Y}{U}(s)\right]$ on occasion during this report for the purpose of compactness and because it is not realistic to define a division process in matrix notation. It is also found that

$$G'\left[-Is-F'\right]^{-1}H' = W_x(s) = \left[\frac{Y}{U}(-s)\right]' \tag{3-28}$$

is the transpose of Equation 3-27 with s replaced by -s.

In other words,

$$G'\left[-Is-F'\right]^{-1}H' = \begin{array}{c} \text{SAME INPUT VARIABLE} \longrightarrow \\ \left[\begin{array}{cccc} \frac{y_1}{u_1}(-s) & \frac{y_2}{u_1}(-s) & \cdots & \frac{y_n}{u_1}(-s) \\ \frac{y_1}{u_2}(-s) & \frac{y_2}{u_2}(-s) & \cdots & \frac{y_n}{u_2}(-s) \\ \vdots & \vdots & & \vdots \\ \frac{y_1}{u_p}(-s) & \frac{y_2}{u_p}(-s) & \cdots & \frac{y_n}{u_p}(-s) \end{array}\right] \end{array} \tag{3-29}$$

(SAME OUTPUT VARIABLE ↓)

So it can be seen that the matrix form of the multivariable root square locus is very similar to the form described by Chang (Reference 3) for the scalar, single-input, single-output case.

Several short examples will demonstrate the computations involved in the root square locus expression.

Single-Input, Single-Output System

Consider the system described simply by the transfer function

$$\frac{y}{u}(s) = w(s) = \frac{N}{D}(s)$$

and the design criterion, or performance index

$$2V = \int_0^\infty (qy^2 + ru^2)\,dt$$

where y = the output of the system
u = the input to the system
q = a scalar, the weighting parameter for the output
r = a scalar, the weighting parameter for the input

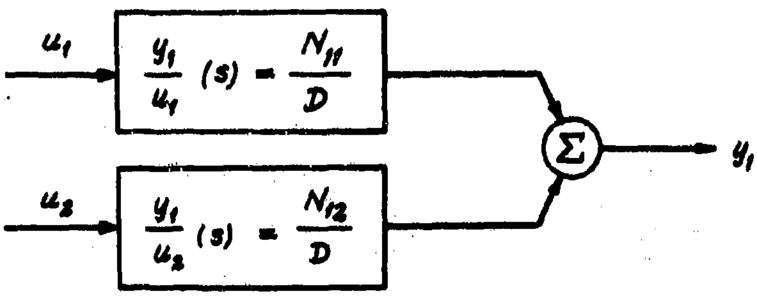
Substituting into the expression for the root square locus yields

$$1 + \frac{q}{r}\frac{y}{u}(-s)\frac{y}{u}(s) = 1 + \frac{q}{r}\frac{N}{D}(-s)\frac{N}{D}(s) = 0 \tag{3-30}$$

This is of exactly the same form as developed in Reference 3.

Single-Output, Dual-Input System

Consider the single-output, dual-input system as shown in the block diagram below:



Let the performance index be

$$2V = \int_0^\infty (q\,y_1^2 + r_1 u_1^2 + r_2 u_2^2)\,dt$$

The expression for the root square locus becomes:

$$\left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r_1^{-1} & 0 \\ 0 & r_2^{-1} \end{bmatrix} \begin{bmatrix} \frac{y_1}{u_1}(-s) \\ \frac{y_1}{u_2}(-s) \end{bmatrix} \begin{bmatrix} q \end{bmatrix} \begin{bmatrix} \frac{y_1}{u_1}(s) & \frac{y_1}{u_2}(s) \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} 1 + q r_1^{-1} \frac{y_1}{u_1}(-s)\frac{y_1}{u_1}(s) & q r_1^{-1} \frac{y_1}{u_1}(-s)\frac{y_1}{u_2}(s) \\ q r_2^{-1} \frac{y_1}{u_2}(-s)\frac{y_1}{u_1}(s) & 1 + r_2^{-1} q \frac{y_1}{u_2}(-s)\frac{y_1}{u_2}(s) \end{vmatrix} = 0$$

$$1 + q r_1^{-1} \frac{y_1}{u_1}(-s)\frac{y_1}{u_1}(s) + q r_2^{-1} \frac{y_1}{u_2}(-s)\frac{y_1}{u_2}(s) = 0$$

(3-31)

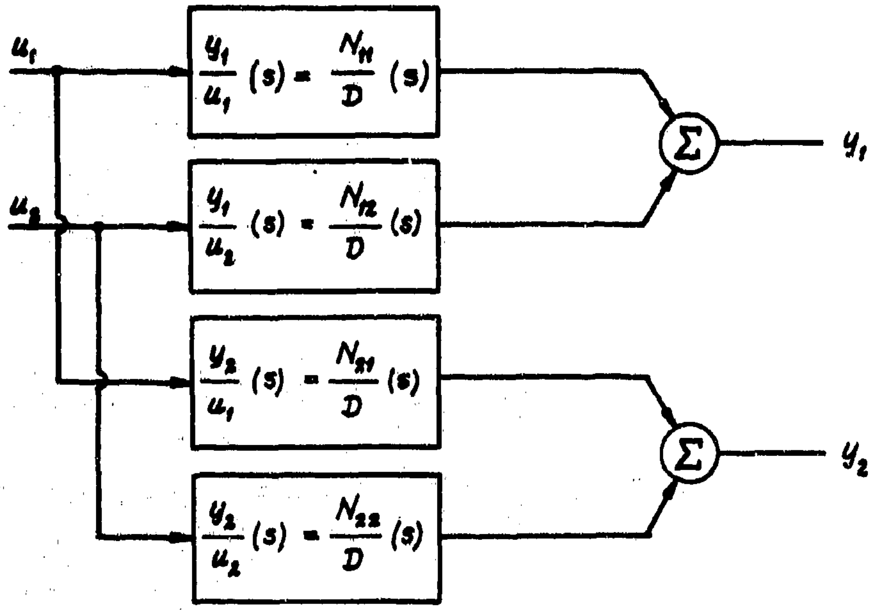
It can be seen that the closed-loop poles of the optimal system and its adjoint are a function of two parameters, q/r_1 and q/r_2. Equation 3-31 can be written

$$0 = 1 + \frac{q}{r_1} \frac{y_1}{u_1}(-s)\frac{y_1}{u_1}(s) \left[1 + \frac{r_1}{r_2} \frac{\frac{y_1}{u_2}(-s)\frac{y_1}{u_2}(s)}{\frac{y_1}{u_1}(-s)\frac{y_1}{u_1}(s)} \right]$$

Two individual root square loci are required, with parameters q/r_1 and r_1/r_2 .

Two-Input, Two-Output System

As a final example, consider the two-input, two-output system as shown below.



with the design performance criterion

$$2V = \min_u \int_0^\infty (y'Qy + u'Ru)\,dt$$

$$= \min_u \int_0^\infty (q_1 y_1^2 + q_2 y_2^2 + r_1 u_1^2 + r_2 u_2^2)\,dt$$

Substituting in Equation 3-26 as before the root square locus expression becomes

$$\left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{D(s)D(-s)} \begin{bmatrix} r_1^{-1} & 0 \\ 0 & r_2^{-1} \end{bmatrix} \begin{bmatrix} N_{11}(-s) & N_{21}(-s) \\ N_{12}(-s) & N_{22}(-s) \end{bmatrix} \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} 1 + \dfrac{q_1 r_1^{-1} N_{11}\bar{N}_{11}}{D\bar{D}} + \dfrac{q_2 r_1^{-1} N_{21}\bar{N}_{21}}{D\bar{D}} & \dfrac{q_1 r_1^{-1} \bar{N}_{11} N_{12}}{D\bar{D}} + \dfrac{q_2 r_1^{-1} \bar{N}_{21} N_{22}}{D\bar{D}} \\ \dfrac{q_1 r_2^{-1} \bar{N}_{12} N_{11}}{D\bar{D}} + \dfrac{q_2 r_2^{-1} \bar{N}_{22} N_{21}}{D\bar{D}} & 1 + \dfrac{q_1 r_1^{-1} \bar{N}_{12} N_{12}}{D\bar{D}} + \dfrac{q_2 r_2^{-1} N_{22}\bar{N}_{22}}{D\bar{D}} \end{vmatrix} = 0$$

which expands to:

$$1 + q_1 r_1^{-1} \frac{N_{11}\bar{N}_{11}}{D\bar{D}} + q_2 r_1^{-1} \frac{\bar{N}_{21} N_{21}}{D\bar{D}} + q_1 r_2^{-1} \frac{N_{12}\bar{N}_{12}}{D\bar{D}} + q_2 r_2^{-1} \frac{N_{22}\bar{N}_{22}}{D\bar{D}}$$

$$+ \frac{1}{(D\bar{D})^2} \begin{vmatrix} r_1^{-1} & 0 \\ 0 & r_2^{-1} \end{vmatrix} \begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} \begin{vmatrix} \bar{N}_{11} & \bar{N}_{21} \\ \bar{N}_{12} & \bar{N}_{22} \end{vmatrix} = 0 \tag{3-32}$$

It would appear from Equation 3-32 that the last term, with $1/(D\bar{D})^2$, would produce a puzzling situation, where the number of closed-loop roots would be doubled. It has been shown, however, that the expression for the root square locus is derived from Equation 3-23, which contains only 2n roots, those of the optimal system and its adjoint. It must be then that a factor $D\bar{D}$ is common to both the numerator and the denominator of the last term of Equation 3-32. Such a factor is in fact common to both numerator and denominator, and this can be shown by an expansion of minors of the original F matrix from which the transfer functions were obtained.

Consider the last term in Equation 3-32, which can be written

$$\frac{q_1 q_2 r_1^{-1} r_2^{-1}}{(D\bar{D})^2} (N_{11}N_{22} - N_{12}N_{21})(\bar{N}_{11}\bar{N}_{22} - \bar{N}_{12}\bar{N}_{21})$$

where

$$N_{11}N_{22} - N_{12}N_{21} = \left| H\,[Is-F]^{adj} G \right| \tag{3-33}$$

and where the H, G and F matrices come from the original equations of motion

$$\dot{x} = Fx + Gu \qquad y = Hx \tag{3-34}$$

If Equation 3-34 is of second order and there are two inputs and two outputs, there appears in the root square locus

$$(N_{11}N_{22} - N_{12}N_{21}) = \left| H\,[Is-F]^{adj} G \right| = |H| \left| [Is-F]^{adj} \right| |G| \tag{3-35}$$

(H and G must be square)

Equation 3-35 can be written

$$(N_{11}N_{22} - N_{12}N_{21}) = |H| \, |Is-F|^{n-1} \, |G| \tag{3-36}$$

since $\left|[Is-F]^{adj}\right| = |Is-F|^{n-1}$ (Reference 14, page 42).

For a second-order system, n = 2 and Equation 3-36 is reducible to

$$(N_{11}N_{22} - N_{12}N_{21}) = D\,|H||G| \tag{3-37}$$

so that the last term in Equation 3-32 becomes

$$\frac{q_1 q_2 r_1^{-1} r_2^{-1}(N_{11}N_{22}-N_{12}N_{21})(\bar{N}_{11}\bar{N}_{22}-\bar{N}_{12}\bar{N}_{21})}{(D\bar{D})^2} = \frac{q_1 q_2 r_1^{-1} r_2^{-1}|H|^2|G|^2 D\bar{D}}{(D\bar{D})^2}$$

$$= \frac{q_1 q_2 r_1^{-1} r_2^{-1}|H|^2|G|^2}{D\bar{D}} \tag{3-38}$$

If H and G are not square, the system reduces to a single-input, single-output problem and the term in the root square locus given by Equation 3-38 does not exist.

For the vast majority of systems, G and H are not square, but the same kind of cancellation has been shown to occur in every example that has been tried. As an example, consider the fourth-order system:

$$[Is-F]^{adj} = \begin{bmatrix} A_{11} & -A_{21} & A_{31} & -A_{41} \\ -A_{12} & A_{22} & -A_{32} & A_{42} \\ A_{13} & -A_{23} & A_{33} & -A_{43} \\ -A_{14} & A_{24} & -A_{34} & A_{44} \end{bmatrix} \qquad G = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & 0 \\ g_{41} & g_{42} \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3-39}$$

where A_{ij} = the minors of a_{ij} in $|Is - F| = A$; i^{th} row, j^{th} column deleted.

The "transfer function" or weighting function matrix $H[Is-F]^{-1}G$ becomes:

$$\frac{1}{|Is-F|}\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{g_{11}A_{13}+g_{31}A_{33}-g_{41}A_{43}}{|Is-F|} & \dfrac{g_{12}A_{13}-g_{42}A_{43}}{|Is-F|} \\ \dfrac{-g_{11}A_{14}-g_{31}A_{34}+g_{41}A_{44}}{|Is-F|} & \dfrac{-g_{12}A_{14}+g_{42}A_{44}}{|Is-F|} \end{bmatrix} \tag{3-40}$$

and the determinant of the above matrix becomes

$$\frac{1}{|Is-F|}\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \frac{1}{|Is-F|^2}\Big\{(g_{12}g_{41}-g_{11}g_{42})(A_{14}A_{43}-A_{13}A_{44})$$

$$+ g_{12}g_{31}(A_{13}A_{34}-A_{14}A_{33}) - g_{31}g_{42}(A_{34}A_{43}-A_{33}A_{44})\Big\} \tag{3-41}$$

Using the determinant identity (see for instance Network Analysis and Feedback Amplifier Design, by Bode, page 54).

$$|A| A_{ab,cd} = A_{ab} A_{cd} - A_{ad} A_{cb}$$

where $|A|$ is any determinant

a, c are any two deleted rows of $|A|$

b, d are any two deleted columns of $|A|$

$A_{ab,cd}$ is called the second minor of $|A|$

A_{ij} is called the first minor of $|A|$

Equation 3-41 becomes:

$$\frac{1}{|Is-F|^2}\begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} = \frac{1}{|Is-F|^2}\Big\{|Is-F|\,\big[g_{12}\,g_{31}\,A_{13,34} - g_{31}\,g_{42}\,A_{34,43}$$

$$+ (g_{12}\,g_{41} - g_{11}\,g_{42})\,A_{14,43}\big]\Big\}$$

$$= \frac{1}{|Is-F|}\Big\{g_{12}\,g_{31}\,A_{13,34} - g_{31}\,g_{42}\,A_{34,43} + (g_{12}g_{41} - g_{11}\,g_{42})A_{14,43}\Big\} \tag{3-42}$$

This proves that for this example, $|Is - F|$ cancels out of the last term in the root square locus. Appendix II describes in more detail the proof that $|Is - F||-Is - F'|$ is common to the quadratic terms in the root square locus expression, showing that the quadratic terms can be obtained from an expansion of minors after the columns of the G matrix have replaced certain columns of the F matrix.

3.3 NUMERICAL EXAMPLE - ROOT SQUARE LOCUS

Consider the aircraft equations of motion given by:

$\Delta\ddot{\theta} = M_{\dot{\theta}}\Delta\dot{\theta} + M_{\alpha}\,\Delta\alpha + M_{\dot{\alpha}}\,\Delta\dot{\alpha} + M_{\delta_e}\,\Delta\delta_e$	Pitch Acceleration Equation	
$\Delta\dot{\theta} - \Delta\dot{\alpha} = L_{\alpha}\,\Delta\alpha + L_{\delta_e}\,\Delta\delta_e$	Equation defining rate of change of flight path	(3-43)
$\tau\,\Delta\dot{\delta}_e + \Delta\delta_e = \Delta\delta_c$	Actuator Dynamics	

In first-order form, with $\dot{\theta}$, α and δ_e state variables, the equations of motion become

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\ddot{\theta} \\ \Delta\dot{\delta}_e \end{bmatrix} = \begin{bmatrix} -L_{\alpha} & 1 & -L_{\delta_e} \\ M_{\alpha} - M_{\dot{\alpha}}L_{\alpha} & M_{\dot{\theta}} + M_{\dot{\alpha}} & M_{\delta_e} - M_{\dot{\alpha}}L_{\delta_e} \\ 0 & 0 & -1/\tau \end{bmatrix}\begin{bmatrix} \Delta\alpha \\ \Delta\dot{\theta} \\ \Delta\delta_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/\tau \end{bmatrix}\Delta\delta_c \tag{3-44}$$

Using the equations of motion typical of a modern, high-performance fighter aircraft, these derivatives can be, for example,

$$M_{\alpha} = -.742 \text{ sec}^{-2} \qquad L_{\alpha} = .535 \text{ sec}^{-1}$$

$$M_{\dot{\theta}} = -.257 \text{ sec}^{-1} \qquad L_{\delta_e} = .109 \text{ sec}^{-1}$$

$$M_{\dot{\alpha}} = -.267 \text{ sec}^{-1} \qquad \tau = .1 \text{ sec}$$

$$M_{\delta_e} = -2.08 \text{ sec}^{-2}$$

Consider the performance criterion given by

$$2V = \min_{u} \int_0^{\infty} (q\alpha^2 + r\,\delta_e^{\,2})dt \qquad (3\text{-}45)$$

It is desired to obtain the closed-loop characteristics of the optimal system that results when the integral of Equation 3-45 is used as a criterion. The closed-loop roots of the optimal system together with the adjoint system can be determined by the root square locus

$$\left| I + R^{-1}G'\left[-Is-F'\right]^{-1}H'QH\left[Is-F\right]^{-1}G \right| = 0 \qquad (3\text{-}46)$$

where

$$H\left[Is-F\right]^{-1}G = \frac{1}{D(s)}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$= \frac{10}{D(s)} A_{31} = \frac{-1.09\,(s+19.3)}{(s+10)(s^2+.944s+.874)} = \frac{\Delta\alpha}{\Delta\delta_a}(s)$$

where A_{31} is the first minor obtained from $|Is - F|$ by deleting the third row and first column and

$$D(s) = |Is-F|$$

Similarly,

$$G'\left[-Is-F'\right]^{-1}H' = \frac{-1.09\,(-s+19.3)}{(-s+10)(s^2-.944s+.874)} = \frac{\Delta\alpha}{\Delta\delta_a}(-s)$$

Substituting into the root square locus expression yields

$$1 + \frac{q}{r}\,\frac{\alpha}{\delta_a}(s)\,\frac{\alpha}{\delta_a}(-s) = 0$$

or

$$-.174 = \frac{q}{r}\;\frac{\left(\pm\dfrac{s}{19.3}+1\right)}{\left(\pm\dfrac{s}{10}+1\right)\left[\left(\dfrac{s}{.935}\right)^2 \pm \dfrac{2(.505)}{.935}\,s+1\right]} \qquad (3\text{-}47)$$

The actual plot of the locus of the closed-loop poles of the optimal system is shown in Figure 3. Although the open-loop adjoint poles and zeros are shown, the locus of the closed-loop adjoint system is omitted for purposes of clarity. The plot has been performed on an ESIAC root locus plotter, the ordinate is phase angle (or damping ratio) and the abscissa is $|s|$. The "fish scales" represent constant values of the real or the imaginary part of the Laplace variable s. The parameter of the locus is q/r, the ratio of the weighting factors of $\Delta\alpha$ and $\Delta\delta_e$. It can be seen from the plot that as q/r is increased from zero, the poles of the closed-loop optimal system appear to approach a damping ratio of $\zeta = .707$ in the limit. In fact, it can be shown that the excess poles over zeros of the root square locus approach a Butterworth pattern as the q/r becomes large (Reference 3). It has been found that the approximation to a Butterworth is good for even small q/r values.

The root square locus plot illustrated in this example supports the intuitive basis for the selection of the matrices Q and R in the performance index. One may select Q and R to trade off control deflection magnitudes for speed of response. The root square locus plot demonstrates that a systematic change in the selection of Q and R results in a gradual, predictable change of the dynamic characteristics of the system. Regardless of the values of Q and R chosen, the optimal system will tend to have smooth, well behaved transient characteristics. It is believed that the dynamic response of a linear optimal system frequently is characteristic of the type of response that one intuitively tries to attain when using conventional design procedures.

He also wishes to determine the feedback gains required to obtain the closed-loop dynamics. These feedback gains can be easily found. The optimal control law is given by

$$u_0 = -R^{-1}G'Px = -Kx$$

and the optimal closed-loop regulator becomes

$$\dot{x} = (F - GK)x$$

whose characteristic equation is

$$|Is - F + GK| = 0 \tag{3-48}$$

or, in terms of the aerodynamic derivatives:

$$\left| \begin{bmatrix} s+.535 & -1 & .109 \\ .599 & s+.524 & 2.05 \\ 0 & 0 & s+10 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} s+.535 & -1 & .109 \\ .599 & s+.524 & 2.05 \\ 10k_1 & 10k_2 & s+10+10k_3 \end{vmatrix} = 0 \tag{3-49}$$

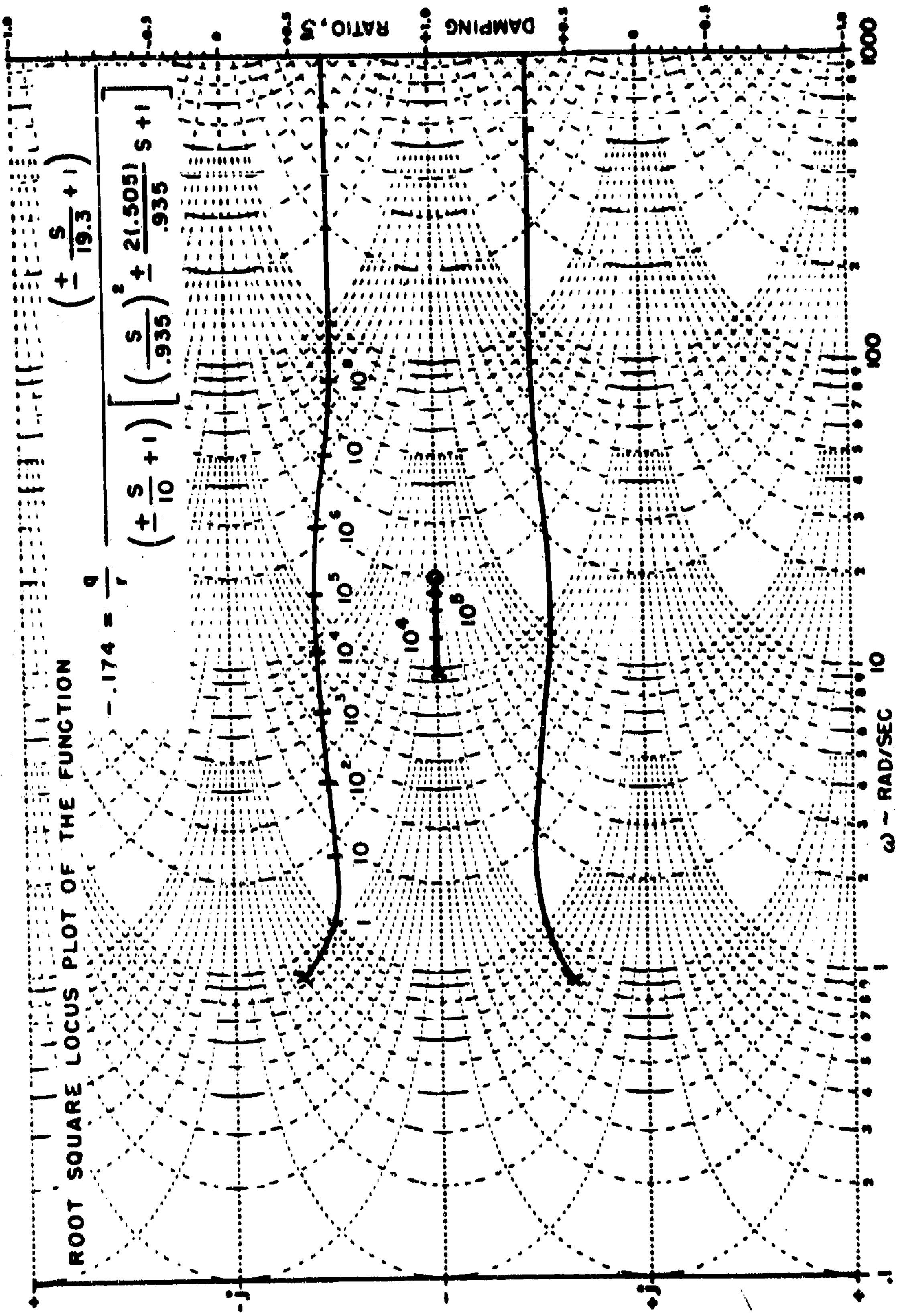


Figure 3. Root Square Locus Plot - Single Input Example

The expansion of Equation 3-49 yields the characteristic equation for the closed-loop optimal system in terms of the feedback gains.

Expanding, there results

$$s^3 + s^2(11.06 + 10k_3) + s(11.47 + 10.59k_3 - 20.5k_2 - 1.09k_1) + (8.79 + 8.79k_3 - 10.3k_2 - 21.1k_1) = 0 \tag{3-50}$$

From the root locus plot, Figure 3, a polynomial of the closed-loop characteristic equation can be formed. As an example, if q/r = 10 were chosen, it is found from the plot that the closed-loop roots are obtained from the characteristic equation

$$(s + 10.2)\left[s^2 + 2(.69)(2.55)s + (2.55)^2\right] = 0$$

or

$$s^3 + 13.72s^2 + 42.4s + 66.3 = 0 \tag{3-51}$$

Equating powers of s of Equations 3-50 and 3-51, the feedback gains are found to be obtained from the solutions of the three equations:

$$\begin{aligned} 11.06 + 10k_3 &= 13.72 \\ 11.47 + 10.59k_3 - 20.5k_2 - 1.09k_1 &= 42.4 \\ 8.79 + 8.79k_3 - 10.3k_2 - 21.1k_1 &= 66.3 \end{aligned} \tag{3-52}$$

Solving these equations yields the feedback gains

$$\begin{aligned} k_3 &= 0.266 \\ k_2 &= -1.265 \\ k_1 &= -2.00 \end{aligned} \tag{3-53}$$

The closed-loop flow diagram is

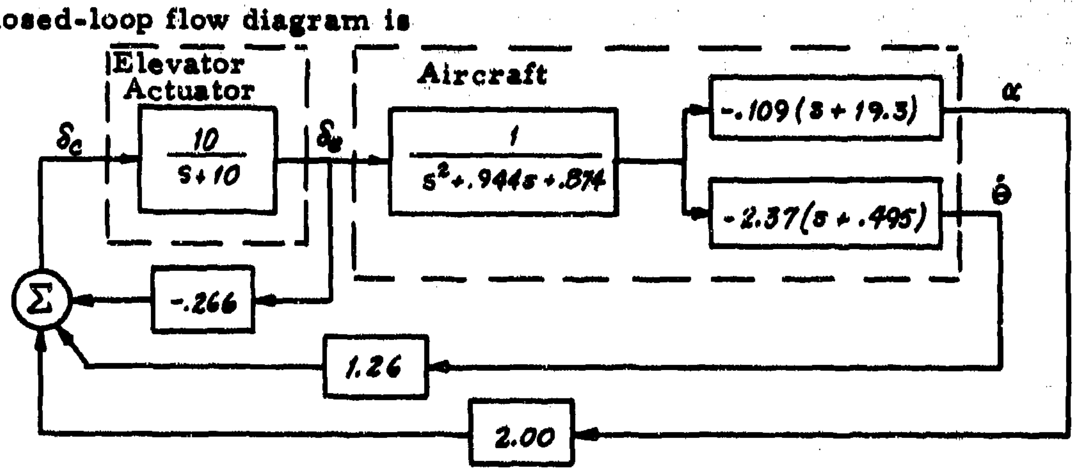


Figure 4. Flow Diagram of Optimal Regulator

It has been argued that optimal systems are impractical because of the multiplicity of feedback gains required. However, it can be easily seen that because the system is assumed to be completely linear, the state variables are related by transfer functions. For instance, $\Delta\dot{\theta}$ can always be reconstructed from $\Delta\alpha$ with a lead-lag network, eliminating the requirement for separate $\Delta\alpha$ and $\Delta\dot{\theta}$ sensors. Also, because parts of the system are at the designer's disposal, the feedback can often be incorporated in an altered design. For instance, the feedback $\Delta\delta_c = -.266\,\Delta\delta_e$ simply represents a requirement for an actuator with a time constant different from $\tau = .1$ seconds.

The above example is simple and could have been solved by conventional techniques, but the example does show that an optimal control approach gives an orderly treatment of problems. The optimal control approach remains orderly for any system complexity.

3.4 ROOT SQUARE LOCUS – CONTROL RATE IN THE PERFORMANCE INDEX

It was seen, from Equation 3-8, that a variety of quadratic forms can be used in connection with the quadratic performance index. Consider the performance index that contains not only the square of the control but also the square of the control rate. It has been shown previously (Reference 2) that an acceptable trade-off between state variable excursions can be obtained for the optimal regulator by trial and error techniques. If the control deflections are greater than desired, say, to avoid amplitude saturation, it is necessary only to increase the relative values of R appearing in the performance index. It is felt that the inclusion of a $\dot{u}$ quadratic term in the performance index will enable the control system designer to influence the relative control deflection rates of the optimal solution. In addition, it will be shown that this addition produces the equivalent of an additional lag term in the root square locus, and therefore, in the optimal system as well.

Consider the problem whereby it is desired to obtain a design satisfying the performance index

$$2V = \int_0^\infty (x'H'QHx + u'Ru + \dot{u}'T\dot{u})\,dt \tag{3-54}$$

subject to the usual constraining equation, the original equations of motion of the system

$$\dot{x} = Fx + Gu \qquad y = Hx$$

The matrices R and T are defined to be symmetrical positive definite matrices weighting respectively the quadratic functions of the control deflection and control deflection rates.

To obtain the Euler-Lagrange equations it is convenient to first generate the function

$$\mathcal{L} = \tfrac{1}{2}\left[x'H'QHx + u'Ru + \dot{u}'T\dot{u}\right] + \lambda'\left[-\dot{x} + Fx + Gu\right] \tag{3-55}$$

where λ is, as before, an undefined column vector which is a function of time.

The Euler-Lagrange equations then become:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0$$

Performing the indicated operations yields:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}(H'QHx + H'QHx) + F'\lambda = H'QHx + F'\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\lambda$$

$$\frac{\partial \mathcal{L}}{\partial u} = \frac{1}{2}(R'u + Ru) + G'\lambda = Ru + G'\lambda \tag{3-56}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{u}} = \frac{1}{2}(T'\dot{u} + T\dot{u}) = T\dot{u}$$

Combining the Euler-Lagrange equations with the constraining equation yields:

$$\begin{aligned} -\dot{x} + Fx + Gu &= 0 \\ H'QHx + \dot{\lambda} + F'\lambda &= 0 \\ -T\ddot{u} + G'\lambda + Ru &= 0 \end{aligned} \tag{3-57}$$

The multivariable root square locus expression is desired for the variational set of equations (3-57). To find the expression for the characteristic equation of the above set of equations, first take the Laplace transform and obtain:

$$\begin{bmatrix} [Is - F] & 0 & -G \\ -H'QH & -[Is + F'] & 0 \\ 0 & -T^{-1}G' & +Is^2 - T^{-1}R \end{bmatrix} \begin{bmatrix} x(s) \\ \lambda(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ -\lambda(0) \\ -su(0) - \dot{u}(0) \end{bmatrix} \tag{3-58}$$

The determinant associated with the matrix set of equations (3-58), when set equal to zero, defines the characteristic polynomial of the optimal system, which includes the stable solution and the adjoint, or image solution.

It will be convenient to use Gauss' Algorithm to find an equivalent expression to the determinant of Equation 3-58. This determinant is of the partitioned form

$$\begin{vmatrix} C_{11} & 0 & C_{13} \\ C_{21} & C_{22} & 0 \\ 0 & C_{32} & C_{33} \end{vmatrix} = 0$$

where C_{ij} is a submatrix within the determinant of a partitioned matrix. The following transformation does not alter the value of the determinant if C_{11} is square and $C_{11} \neq 0$.

$$\begin{vmatrix} C_{11} & 0 & C_{13} \\ C_{21} & C_{22} & 0 \\ 0 & C_{32} & C_{33} \end{vmatrix} = \begin{vmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22}^{(1)} & C_{23}^{(1)} \\ 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{vmatrix}$$

where

$$C_{\alpha\beta}^{(1)} = -C_{\alpha 1}\, C_{11}^{-1}\, C_{1\beta} + C_{\alpha\beta}$$

In terms of the matrices of the determinant of Equation 3-58 the result is

$$\begin{vmatrix} [Is-F] & 0 & -G \\ -H'QH & -[Is+F'] & 0 \\ 0 & -T^{-1}G & +Is^2-T^{-1}R \end{vmatrix} = \begin{vmatrix} [Is-F] & 0 & -G \\ 0 & -[Is+F'] & H'QH[Is-F]^{-1}G \\ 0 & -T^{-1}G' & Is^2-T^{-1}R \end{vmatrix} \qquad (3\text{-}59)$$

Repeating the Algorithm:

$$\begin{vmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22}^{(1)} & C_{23}^{(1)} \\ 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{vmatrix} = \begin{vmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22}^{(1)} & C_{23}^{(1)} \\ 0 & 0 & C_{33}^{(2)} \end{vmatrix}$$

where $C_{33}^{(2)} = -C_{32}^{(1)}\left[C_{22}^{(1)}\right]^{-1} C_{23}^{(1)} + C_{33}^{(1)}$

$$= \left[T^{-1}G'\right]\left[-Is-F'\right]^{-1}\left[H'QH(Is-F)^{-1}G\right] + Is^2-T^{-1}R$$

Equation 3-59 finally becomes

$$0 = \begin{vmatrix} [Is-F] & 0 & -G \\ -H'QH & [-Is-F'] & 0 \\ 0 & -T^{-1}G' & Is^2-T^{-1}R \end{vmatrix} = \begin{vmatrix} [Is-F] & 0 & -G \\ 0 & [-Is-F'] & H'QH[Is-F]^{-1}G \\ 0 & 0 & \begin{matrix}(T^{-1}G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G \\ +Is^2-T^{-1}R)\end{matrix} \end{vmatrix}$$

which can be written as follows:

$$|Is-F||-Is-F'||[Is^2-T^{-1}R]+T^{-1}G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G| = 0 \qquad (3\text{-}60)$$

Because $|Is - F|$ and $|-Is - F'|$ are by definition square and not equal to zero except at the open-loop roots, it is necessary to consider only

$$|[Is^2-T^{-1}R]+T^{-1}G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G| = 0$$

or

$$|I+[Is^2-T^{-1}R]^{-1}T^{-1}G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G| = 0 \qquad (3\text{-}61)$$

to obtain the closed-loop roots of the optimal system and its adjoint. Notice also that for this performance index, $|Is^2-T^{-1}R|$ defines additional open-loop roots.

It can be seen that the $\dot{u}'T\dot{u}$ additions to the performance index produce a slightly different expression for the root square locus, with $[Is^2-T^{-1}R]^{-1}T^{-1}$ replacing R^{-1}. This has the effect of adding additional poles in the root square locus plot at the values indicated by the expression $[Is^2-T^{-1}R]^{-1}$. The effect on the optimal closed-loop system is to add first-order lag networks, one for each input or controller.

3.5 ROOT SQUARE LOCUS - OUTPUT RATES IN THE PERFORMANCE INDEX

Consider the performance index containing only the output rates and the control deflections in the performance index, i.e.,

$$2V = \int_0^\infty (\dot{y}'M\dot{y} + u'Ru)dt \qquad (3\text{-}62)$$

where M and R are defined as positive definite symmetrical matrices weighting the individual terms of the performance index. The constraint is, as usual, the equations of motion

$$\dot{x} = Fx + Gu \qquad (a)$$
$$y = Hx \qquad (b) \qquad (3\text{-}63)$$

Substituting for $\dot{y}$ in Equation 3-62 yields

$$2V = \int_0^\infty (\dot{x}'H'MH\dot{x} + u'Ru)\,dt$$

The Euler equations associated with the solution of this problem are, as before,

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0$$

where

$$\mathcal{L} = \tfrac{1}{2}(\dot{x}'H'MH\dot{x} + u'Ru) + \lambda'\cdot(-\dot{x} + Fx + Gu) \qquad (3\text{-}64)$$

The Euler equations yield

$$\dot{\lambda} + F'\lambda + H'MH\ddot{x} = 0 \qquad (a)$$

$$Ru_0 + G'\lambda = 0, \text{ hence } u_0 = -R^{-1}G'\lambda \qquad (b) \qquad (3\text{-}65)$$

Combining Equations 3-63a and 3-65, the result is, after taking a Laplace transform

$$\begin{bmatrix} [Is-F] & GR^{-1}G' \\ [-Is]H'MH[Is] & [-Is-F'] \end{bmatrix} \begin{bmatrix} x(s) \\ \lambda(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ -H'MH[Isx(0)+\dot{x}(0)]+[\lambda(0)] \end{bmatrix} \tag{3-66}$$

The determinant of Equation 3-66, when set equal to zero, defines the roots of the closed-loop optimal system and its adjoint.

Again, by using Gauss' Algorithm, it can be shown that the root square locus becomes:

$$\left| I + [-Is]R^{-1}[Is]G'[-Is-F']^{-1}H'MH[Is-F]^{-1}G \right| = 0 \tag{3-67}$$

It is clear from Equation 3-67 that zeros are added to the root square locus plot at the origin. Otherwise, the expression is the same as Equation 3-25. Adding zeros at the origin of a root locus plot is equivalent to obtaining the locus of the derivative of a transfer function. Equation 3-67 can then be written

$$\left| I + [-Is]R^{-1}[Is]\left[\frac{Y}{u}(-s)\right]' M \left[\frac{Y}{u}(s)\right] \right| = 0$$

or

$$\left| I + [-Is]R^{-1}[Is]\, W_*(s)\, M W(s) \right| = 0$$

or

$$\left| I + R^{-1} \left[\frac{\dot{Y}}{U}(-s)\right]' M \left[\frac{\dot{Y}}{U}(s)\right] \right| = 0 \tag{3-68}$$

The Riccati equation can be easily obtained for this particular performance index by substituting $\dot{x} = Fx + Gu$ in Equation 3-64 before the Euler equations are obtained. The function $\mathcal{X}$ becomes

$$\mathcal{X} = \frac{1}{2}\left\{(x'F'+u'G')(H'MH)(Fx+Gu)+u'Ru\right\} + \lambda'\cdot(-\dot{x}+Fx+Gu)$$

$$= \frac{1}{2}x'F'H'MHFx + \frac{1}{2}x'F'H'MHGu + \frac{1}{2}u'G'H'MHFx$$

$$+ \frac{1}{2}u'(G'H'MHG+R)u + \lambda'\cdot(-\dot{x}+Fx+Gu)$$

$$\frac{\partial \mathcal{X}}{\partial x} = F'H'MHFx + F'H'MHGu + F'\lambda \qquad \frac{\partial \mathcal{X}}{\partial \dot{x}} = -\lambda$$

$$\frac{\partial \mathcal{X}}{\partial u} = G'H'MHFx + [G'H'MHG+R]u + G'\lambda \qquad \frac{\partial \mathcal{X}}{\partial \dot{u}} = 0$$

The Euler-Lagrange equations, together with the constraining equation, become:

$$\begin{aligned} &\dot{\lambda} + F'\lambda + F'H'MHFx + F'H'MHGu = 0 && \text{(a)} \\ &G'H'MHFx + [G'H'MHG + R]u + G'\lambda = 0 && \text{(b)} \\ &\dot{x} - Fx - Gu = 0 && \text{(c)} \end{aligned} \qquad (3\text{-}69)$$

Solving Equation 3-69b for u yields the optimal control law

$$u_o = -[G'H'MHG + R]^{-1}[G'\lambda + G'H'MHFx] \qquad (3\text{-}70)$$

Now substitute 3-70 into 3-69a and 3-69c and obtain

$$\begin{aligned} &\dot{x} - Fx + G[G'H'MHG + R]^{-1}[G'\lambda + G'H'MHFx] = 0 && \text{(a)} \\ &\dot{\lambda} + F'\lambda + F'H'MHFx - F'H'MHG[G'H'MHG + R]^{-1}[G'\lambda + G'H'MHFx] = 0 && \text{(b)} \end{aligned} \qquad (3\text{-}71)$$

Let $\lambda = Px$, $\dot{\lambda} = P\dot{x}$, where P is a constant matrix, and multiply 3-71a by P, yielding:

$$P\dot{x} - PFx + PG[G'H'MHG + R]^{-1}[G'Px + G'H'MHFx] = 0$$

$$P\dot{x} + F'Px + F'H'MHFx - F'H'MHG[G'H'MHG + R]^{-1}[G'Px + G'H'MHFx] = 0$$

Finally, eliminating $P\dot{x}$ in the above equations yields the Riccati equation

$$P[F - GA^{-1}B] + [F' - B'A^{-1}G']P + [C - B'A^{-1}B] - PGA^{-1}G'P = 0 \qquad (3\text{-}72)$$

where

$A = G'H'MHG + R$
$B = G'H'MHF$
$C = F'H'MHF$

3.6 ROOT SQUARE LOCUS - MODEL IN THE PERFORMANCE INDEX

The use of a mathematical model appearing only in the performance index to describe a desired system matrix was first suggested by Kalman. The technique is as follows. A mathematical model is defined

$$\dot{\eta} = L\eta \qquad (3\text{-}73)$$

where η is some fictitious state variable and L is the system matrix of the desired system. A performance index is specified of the form

$$2V = \int_0^\infty \left([\dot{y} - Ly]'Q[\dot{y} - Ly] + u'Ru\right)dt \qquad (3\text{-}74)$$

It can be seen that if the system output rate $\dot{y}$ behaves exactly as the model, the first term in the performance index, $\dot{y} - Ly$, will be zero. If the control of the system is such that feedback can force the system to behave exactly as the model, the regulator in the limit (i.e., as $|Q|/|R| \rightarrow \infty$) will

behave exactly as the model. To obtain this condition, if it is possible, one need only consider the performance index

$$2V = \int_0^\infty [\dot{y} - Ly]'Q[\dot{y} - Ly]\,dt \tag{3-75}$$

If the model can be exactly matched, an optimal solution will simplify such that the optimal feedback control law becomes a set of fixed feedback gains. It should be pointed out, however, that the case where the system and the model are exactly matchable is the rare exception rather than the rule. In general, it requires that L, F and G be of the same dimension. An artificial procedure would be to have the number of columns of the G matrix equal to the number of non-matching rows of F and L. For instance, if F and L were each obtained from a transfer function, then F and L can be written such that only the entries in the last row of F and L differ. In this case, a single control variable, with a column G matrix, may be sufficient to exactly match the optimal regulator to the model. However, exact model matching in practice often requires large or violent controller motions. The inclusion of the $u'Ru$ term in the performance index allows as close a match as possible within allowable controller motions.

The root square locus for the model technique is obtained in a manner very similar to the previous developments. The performance index is

$$2V = \int_0^\infty \left([\dot{y} - Ly]'Q[\dot{y} - Ly] + u'Ru\right)dt$$

where

$$y = Hx$$
$$\dot{y} = H\dot{x}$$

Substituting for y and $\dot{y}$ yields

$$2V = \int_0^\infty \left([\dot{x}'H' - x'H'L']Q[H\dot{x} - LHx] + u'Ru\right)dt$$

$$= \int_0^\infty \left(\dot{x}'H'QH\dot{x} - \dot{x}'H'QLHx - x'H'L'QH\dot{x} + x'H'L'QLHx + u'Ru\right)dt \tag{3-76}$$

To assure that the integral performance index contains quadratic forms, note that the two bilateral forms $-\dot{x}'H'QLHx$ and $-x'H'L'QH\dot{x}$ constitute the symmetrical part of $-2\dot{x}'H'QLHx$ and therefore can be treated as a quadratic form.

As before, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \qquad \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0$$

where $\mathcal{L} = \frac{1}{2}(\dot{x}'H'QH\dot{x} - \dot{x}'H'QLHx - x'H'L'QH\dot{x} + x'L'H'QLHx + u'Ru) + \lambda' \cdot (-\dot{x} + Fx + Gu)$

The Euler equations become:

$$\dot{\lambda} + F'\lambda - H'QH\ddot{x} + H'QLH\dot{x} - H'L'QH\dot{x} + H'L'QLHx = 0 \quad \text{(a)}$$
$$Ru + G'\lambda = 0 \quad \text{(b)} \qquad (3\text{-}77)$$

Solving Equation 3-77b for u yields the optimal control law

$$u_0 = -R^{-1}G'\lambda$$

The two equations that describe the realizable optimal system and its adjoint are:

$$\begin{aligned} &\dot{x} - Fx + GR^{-1}G'\lambda = 0 \\ &\dot{\lambda} + F'\lambda - H'QH\ddot{x} + H'QLH\dot{x} - H'L'QH\dot{x} + H'L'QLHx = 0 \end{aligned} \qquad (3\text{-}78)$$

In Laplace transform form, these equations become

$$\begin{bmatrix} [Is-F] & GR^{-1}G' \\ -H'[-Is-L']Q[Is-L]H & [-Is-F'] \end{bmatrix} \begin{bmatrix} x(s) \\ \lambda(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ +H'QH[Is\,x(0)+\dot{x}(0)] - H'[QL+L'Q]Hx(0) - \lambda(0) \end{bmatrix} \qquad (3\text{-}79)$$

The determinant of the left-hand side of Equation 3-79, when set to zero, describes the closed-loop poles of the optimal system and its adjoint. Using Gauss' Algorithm, as was done previously in this section, a convenient expression for a root square locus can be obtained:

$$\left| I + R^{-1}G'[-Is-F']^{-1}H'[-Is-L']Q[Is-L]H[Is-F]^{-1}G \right| = 0 \qquad (3\text{-}80)$$

which again is of the general form

$$\left| I + K\frac{N\bar{N}}{D\bar{D}} \right| = 0$$

It can be seen from Equation 3-80 that the elements of the model matrix appear in the root square locus and contribute to the zeros of the root square locus. In fact, the order of the numerator $N\bar{N}$ is generally increased, such that $N\bar{N}$ and $D\bar{D}$ can be of the same order.

If this is the case, the elements of the Q matrix can be chosen such that the zeros of the root square locus are the eigenvalues of the model and adjoint, i.e., $|Is - L|$ and $|-Is - L'|$ and the optimal system will, in the limit, (i.e., as $|Q|/|R|$ becomes large) have closed-loop poles that are identical to the model poles. However, this procedure would in general require some negative values for the elements of the Q matrix. The system can no longer be described as "optimal" as defined in the original formulation of the problem.

However, if one obtains better model following by selecting negative q elements, the distinction between "optimal" and "non-optimal" becomes trivial.

3.7 MODEL AND PLANT EXACTLY MATCHABLE

It is not obvious in the previous formulation of the model performance index problem that the plant will, under certain circumstances, actually match the model exactly. A slightly different formulation of the problem will show the match more clearly.

Starting with the same performance index

$$2V = \int_0^\infty \left([\dot{y} - Ly]' Q [\dot{y} - Ly] + u'Ru \right) dt$$

Substitute for y and $\dot{y}$

$$y = Hx$$
$$\dot{y} = H[Fx + Gu]$$

and obtain

$$\begin{aligned} 2V &= \int_0^\infty \left([HFx + HGu - LHx]' Q [HFx + HGu - LHx] + u'Ru \right) dt \\ &= \int_0^\infty \left(x'[HF - LH]' Q [HF - LH] x + x'[HF - LH]' QHGu \right. \\ &\quad \left. + u'G'H'Q[HF - LH]x + u'[G'H'QHG + R]u \right) dt \end{aligned} \tag{3-81}$$

This performance index is of the quadratic form

$$2V = \int_0^\infty (x'Ax + x'Bu + u'B'x + u'Cu) dt$$

where

$$A = [HF - LH]' Q [HF - LH]$$
$$B = [HF - LH]' QHG$$
$$C = G'H'QHG + R$$

The Euler-Lagrange equations are obtained in the same way as before, using the function

$$\mathcal{L} = \tfrac{1}{2}(x'Ax + x'Bu + u'B'x + u'Cu) + \lambda' \cdot (-\dot{x} + Fx + Gu)$$

The Euler equations are:

$$\left.\begin{aligned} \dot{\lambda}' + F'\lambda + Bu + Ax &= 0 \qquad \text{(a)} \\ B'x + Cu + G'\lambda &= 0 \qquad \text{(b)} \end{aligned}\right\} \tag{3-82}$$

Solving Equation 3-71b,

$$u_o = -C^{-1}B'x - C^{-1}G'\lambda \tag{3-83}$$

The two equations that describe the closed-loop optimal system and its adjoint are Equations 3-82a and the original plant equation, with u_o substituted for u.

$$\begin{aligned} &\dot{x} - Fx + GC^{-1}B'x + GC^{-1}G'\lambda = 0 \\ &\dot{\lambda} + F\lambda + Ax - BC^{-1}B'x - BC^{-1}G'\lambda = 0 \end{aligned} \tag{3-84}$$

After writing these equations in Laplace transformed matrix form, there results:

$$\begin{bmatrix} [Is - F + GC^{-1}B'] & GC^{-1}G' \\ -A + BC^{-1}B' & -[Is + F' - BC^{-1}G'] \end{bmatrix} \begin{bmatrix} x(s) \\ \lambda(s) \end{bmatrix} = \begin{bmatrix} x_o \\ -\lambda_o \end{bmatrix} \tag{3-85}$$

It will be recalled it was stated that if the plant and the model can be exactly matched, the dimension of F and L must be the same, and that, in general, the number of columns of G must equal the number of rows of F and L. Also, the $u'Ru$ term in the performance index can be dropped. Therefore, let R = 0, H = I and dim. F = dim. L. Then HG is invertible and the matrix entries in Equation 3-85 become:

$$\begin{aligned} Is - F + GC^{-1}B' &= Is - F + G[G'H'QHG]^{-1}G'H'Q[HF - LH] \\ &= Is - F + G[G'QG]^{-1}G'Q[F - L] \\ &= Is - L \\ -[Is + F' - BC^{-1}G'] &= -[Is + F' - [HF - LH]'QHG[G'H'QHG]^{-1}G'] \\ &= -Is - L' \\ A - BC^{-1}B' &= 0 \\ GC^{-1}G' &= I \end{aligned}$$

Substituting these values into Equation 3-74, there results

$$\begin{bmatrix} Is - L & I \\ 0 & -Is - L' \end{bmatrix} \begin{bmatrix} x(s) \\ \lambda(s) \end{bmatrix} = \begin{bmatrix} x_o \\ -\lambda_o \end{bmatrix} \tag{3-86}$$

The determinant of this set of equations, when set equal to zero, is

$$|Is - L||-Is - L'| = 0$$

Therefore, the optimal regulator has exactly the same characteristic equation as the model. The optimal regulator satisfies the set of equations

$$\dot{y} = Ly$$

and the model dynamic characteristics have been exactly matched. Under these

circumstances, the optimal feedback control law is given by

$$u_0 = -C^{-1}B'x = -[F-L]x \tag{3-87}$$

When the plant and the model are exactly matchable, optimal techniques are obviously not required to obtain the feedback law which produces the match. The importance of this technique lies in the fact that if the feedback gain magnitudes were restricted, linear optimal techniques will enable the designer to match the model as closely as possible in the integral error squared sense within the allowable control motions. It has been repeatedly demonstrated that a near match in this manner produces a well behaved, smooth approximation to the desired model system matrix.

Two examples will serve to demonstrate the model-in-the-performance index technique under the circumstances when the plant and model are matchable and not matchable.

Example 1: Model and Plant Matchable

Let the system be described by the first-order set of differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f_{21} & -f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g_{21} \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3-88}$$

and let the model be described by the fictitious set of first-order linear equations

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\ell_{21} & -\ell_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{3-89}$$

Notice that the plant and the model vector describe an orthogonal set and the matrices differ in only one row. There is one controller, so it can be hypothesized that these sets of equations can be exactly matched as $|Q|/|R|$ becomes large.

Equation 3-80 is the expression for the root square locus for this problem.

$$|I + R^{-1}G'[-Is-F']^{-1}H'[-Is-L']Q[Is-L]H[Is-F]^{-1}G| = 0 \tag{3-90}$$

It has been shown previously that the ratio $|Q|/|R|$ is the main parameter of the root square locus plot rather than the absolute values of Q or R. For convenience, then, for this example choose

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} r \end{bmatrix}$$

Calculating the entries in the root square locus expression:

$H[Is-F]^{-1}G =$ the generalized transfer function matrix

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+f_{22} & 1 \\ -f_{21} & s \end{bmatrix} \begin{bmatrix} 0 \\ g_{21} \end{bmatrix} \frac{1}{D_F(s)}$$

$$= \frac{g_{21}}{D_F(s)} \begin{bmatrix} 1 \\ s \end{bmatrix} \quad \text{where} \quad D_F(s) = s^2 + sf_{22} + f_{21}$$

Similarly,

$$G'[-Is-F']^{-1}H = \frac{g_{21}}{D_F(-s)} \begin{bmatrix} 1 & -s \end{bmatrix} \quad \text{where} \quad D_F(-s) = s^2 - f_{22}s + f_{21}$$

$$[Is+L']Q[Is-L] = \begin{bmatrix} -s^2+\ell_{21}^2 & s+\ell_{21}(s+\ell_{22}) \\ -s+\ell_{21}(-s+\ell_{22}) & 1+(s-\ell_{22})(-s-\ell_{22}) \end{bmatrix}$$

Substituting into the root square locus expression yields

$$\left| 1 + \frac{r^{-1}g_{21}^2}{D_F(s)D_F(-s)} \begin{bmatrix} 1 & -s \end{bmatrix} \begin{bmatrix} -s+\ell_{21}^2 & s+\ell_{21}(s+\ell_{22}) \\ -s+\ell_{21}(-s+\ell_{22}) & 1+(s-\ell_{22})(-s-\ell_{22}) \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \right| = 0$$

or

$$1 + \frac{g_{21}^2}{r} \frac{(s^2+\ell_{22}s+\ell_{21})(s^2-\ell_{22}s+\ell_{21})}{D_F(s)\,D_F(-s)} = 0$$

or

$$1 + \frac{g_{21}^2}{r} \frac{D_L(s)\,D_L(-s)}{D_F(s)\,D_F(-s)} = 0 \tag{3-91}$$

where $\quad D_L(s) = s^2 + \ell_{22}s + \ell_{21} \;, \quad D_L(-s) = s^2 - \ell_{22}s + \ell_{21}$

Because the zeros of the root square locus $D_L(s)$ and $D_L(-s)$ are the characteristic roots of the model and its adjoint, the root square locus will originate at the poles of the plant and terminate at the model roots. The optimal regulator and the model will be identical as $1/r$ approaches infinity. The feedback gains will, of course, be finite.

To provide an actual numerical example,* let

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 \\ -25 & -7.07 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The expression for the root square locus becomes

$$1 + \frac{r^{-1}(s^2 + 7.07s + 25)(s^2 - 7.07s + 25)}{(s^2 + 2s + 1)(s^2 - 2s + 1)} = 0 \qquad (3\text{-}92)$$

Figure 5 is a root square locus plot performed on an ESIAC root locus plotter, which semiautomatically obtains a root locus on a rectangular plot, with damping ratio or phase angle of s as the ordinate and absolute value of s as the abscissa.

The parameter of the locus is $1/r$ and the plot shows that the plant poles migrate to the model poles rapidly and in an orderly fashion.

The optimal control law, in the limit as $1/r \rightarrow \infty$ is

$$u_0 = -(F - L)x$$

$$= -\left\{ \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -25 & -7.07 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -24 & -5.07 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$u_0 = -24x_1 - 5.07x_2$$

---

* This example was used previously by Tyler, Reference 12.

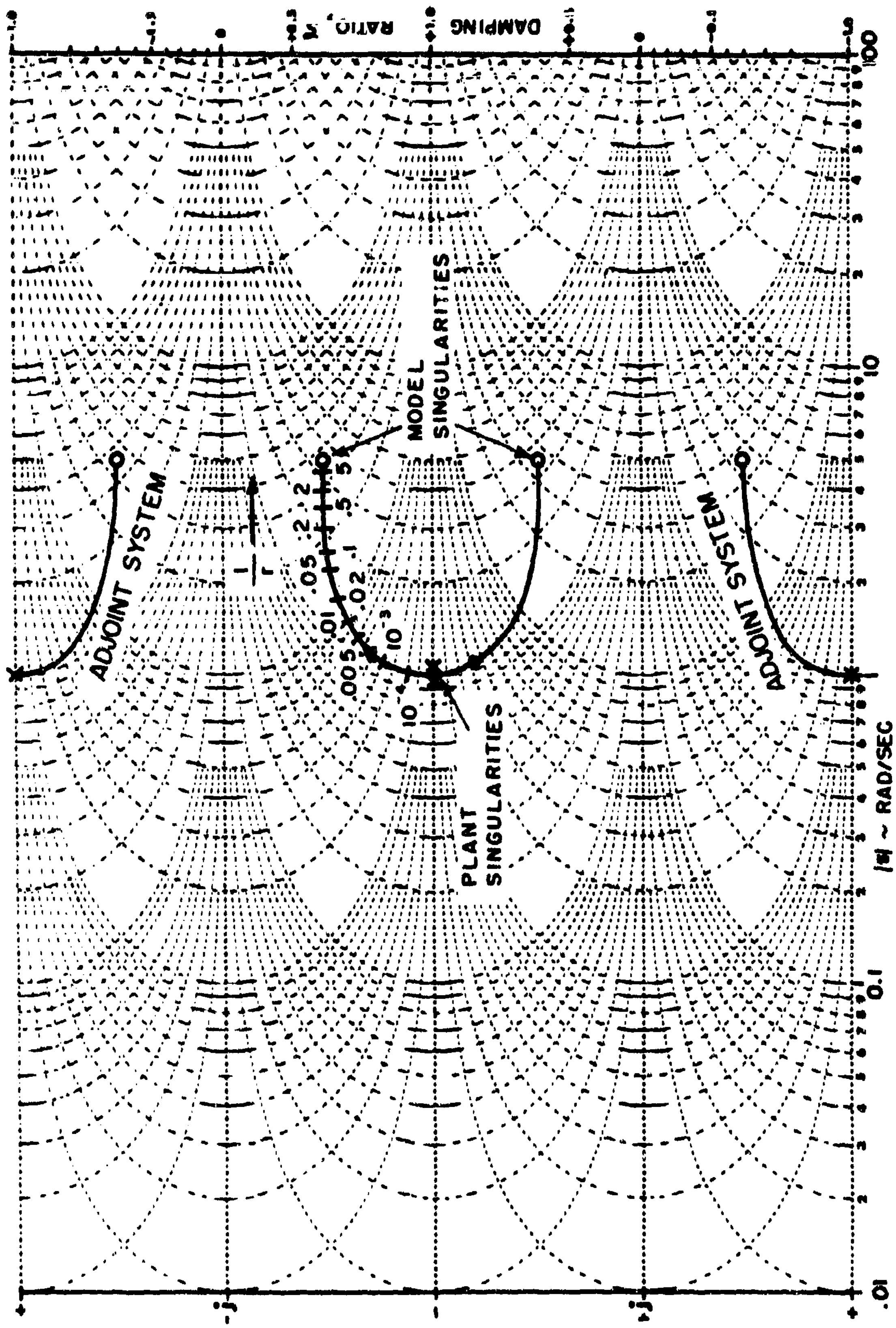


Figure 5. Root Square Locus – Model in the Performance Index

Example 2: Model and Plant Cannot be Matched Identically

As a second example of the use of a model associated with optimal control, it was decided to use a somewhat more realistic system. The simplified equations of longitudinal motion of a high-performance military fighter aircraft were hypothesized for the plant. A model system matrix was specified whose natural frequency and damping are such to satisfy acceptable flying qualities. Natural frequency and damping are, of course, only two parameters of many that are used to define acceptable flying qualities. Other parameters, such as the slope of C_L vs. α curve are equally important.

However, the model-performance index problem as it is presently formulated allows only for a change in the system matrix, i.e., it is possible to change only the state transition matrix. The transfer function matrix is made up of the output matrix and the control matrix as well, so the transfer function of the closed-loop optimal system can be affected only to a limited degree.

If the above logic is extended to the model situation, it should be noted that the performance index for the model case is specified in terms of error rates between a desired response and an actual response. Since whatever appears in the performance index approaches a Butterworth response, it is clear that the error rates between the model and the aircraft will be smooth and well behaved, approaching zero as the weighting on the error is made very large with respect to the control.

It is not possible in this case to exactly match the plant to the model with finite values of error and control weighting. Assuming that the important variables are angle of attack and pitch rate, it will be instructive to see how the closed-loop roots of the optimal regulator behave as a function of the relative weighting of the $\Delta\dot{\theta}$ and $\Delta\alpha$ error rates and the ratio of the error rates and the control deflections.

Assuming straight and level flight, the short period equations of motion about the stability axes can be written:

$$\Delta\ddot{\theta} = M_\alpha \Delta\alpha + M_{\dot{\theta}}\Delta\dot{\theta} + M_{\dot{\alpha}}\Delta\dot{\alpha} + M_{\delta_e}\Delta\delta_e \qquad \text{pitching acceleration equation} \qquad \text{(a)}$$

$$\Delta\dot{\theta} - \Delta\dot{\alpha} = L_\alpha \Delta\alpha + L_{\delta_e}\Delta\delta_e \qquad \text{flight path velocity equation} \qquad \text{(b)} \quad (3\text{-}93)$$

The elevator actuator dynamics were assumed to be

$$\Delta\ddot{\delta}_e = -2\zeta\omega_{ACT}\Delta\dot{\delta}_e - \omega_{ACT}^2\Delta\delta_e + \omega_{ACT}^2\Delta\delta_c$$

These equations of motion are easily put into first-order form by solving for $\Delta\dot{\alpha}$ and substituting for $\Delta\dot{\alpha}$ in the pitching acceleration equation, Equation 3-93b.

In matrix form, the first-order differential equations of motion become

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\ddot{\theta} \\ \Delta\dot{\delta}_e \\ \Delta\ddot{\delta}_e \end{bmatrix} = \begin{bmatrix} -L_\alpha & 1 & -L_{\delta_e} & 0 \\ M_\alpha - M_{\dot{\alpha}} L_\alpha & M_{\dot{\theta}} + M_{\dot{\alpha}} & M_{\delta_e} - M_{\dot{\alpha}} L_{\delta_e} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{ACT}^2 & -2\zeta\omega_{ACT} \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta\dot{\theta} \\ \Delta\delta_e \\ \Delta\dot{\delta}_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega_{ACT}^2 \end{bmatrix} \Delta\delta_c \qquad (3\text{-}94)$$

The model chosen is of the form

$$\begin{bmatrix} \Delta\dot{\alpha}_m \\ \Delta\ddot{\theta}_m \end{bmatrix} = \begin{bmatrix} -L_{\alpha_m} & 1 \\ M_{\alpha_m} - M_{\dot{\alpha}_m} L_{\alpha_m} & M_{\dot{\theta}_m} + M_{\dot{\alpha}_m} \end{bmatrix} \begin{bmatrix} \Delta\alpha_m \\ \Delta\dot{\theta}_m \end{bmatrix} \qquad (3\text{-}95)$$

and the following derivatives have been chosen:

M_α = -.742 sec^{-2} $\qquad$ M_{α_m} = -16.0 sec^{-2}

$M_{\dot{\theta}}$ = -.257 sec^{-1} $\qquad$ $M_{\dot{\theta}_m}$ = -2.9 sec^{-1}

$M_{\dot{\alpha}}$ = -.267 sec^{-1} $\qquad$ $M_{\dot{\alpha}_m}$ = -1.0 sec^{-1}

M_{δ_e} = -2.08 sec^{-2} $\qquad$ L_{α_m} = 1.5 sec^{-1}

L_α = .535 sec^{-1} $\qquad$ ω_{ACT} = 20.0 rad/sec

L_{δ_e} = .109 sec^{-1} $\qquad$ ζ_{ACT} = 1.0

The flight condition associated with the derivatives listed above is one of low speed, low altitude power approach. The short period natural frequency is 0.935 rad/sec and the short period damping ratio is .505. The model has a short period natural frequency of 4.5 rad/sec and a damping ratio of 0.6.

Substituting the aerodynamic derivatives into Equations 3-94 and 3-95 yields, for the F, L and G matrices:

$$F = \begin{bmatrix} -.535 & 1 & -.109 & 0 \\ -.599 & -.524 & -2.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -400 & -40 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 400 \end{bmatrix} \qquad (3\text{-}96)$$

$$L = \begin{bmatrix} -1.5 & 1 \\ -14.5 & -3.9 \end{bmatrix}$$

It is clear that the closed-loop optimal regulator $\dot{x} = (F - GK)x$ cannot be forced to have the dynamics of the model under any circumstances. The aircraft and the model matrices are of different order, and even if L were written as

$$L = \begin{bmatrix} -1.5 & 1 & 0 & 0 \\ -14.5 & -3.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} +\ell_{11} & 1 & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to make the two matrices of the same size, all four rows of F and L are different, and as has been suggested previously, at most, one non-identical row of F and L can be matched with one control variable.

It will be instructive, nevertheless, to investigate the closed-loop dynamics of the optimal system that match the model as closely as possible using the quadratic performance index as a criterion. It will be seen that the closed-loop poles originally associated with the actuator increase in frequency and approach $\zeta = .707$ while the poles originally associated with the short period roots of the aircraft tend toward some frequency and damping intermediate between the frequency and damping of the open-loop aircraft and the model.

Let it be assumed that it is nominally desirable to match the closed-loop aircraft to both $\Delta\alpha_m$ and $\Delta\dot{\theta}_m$. The following H, Q and R matrices can then be chosen:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \qquad R = [r]$$

This yields a performance index

$$2V = \int_0^\infty \left([\dot{y} - Ly]' Q [\dot{y} - Ly] + u' R u \right) dt$$

or

$$2V = \int_0^\infty [q_1(\dot{\alpha} - \ell_{11}\alpha - \ell_{12}\dot{\theta})^2 + q_2(\ddot{\theta} - \ell_{21}\alpha - \ell_{22}\dot{\theta})^2 + r\delta_c^2]dt \tag{3-97}$$

It has been suggested that the matrix H must be the identity matrix [I] for a proper formulation of the model-in-the-performance-index problem. This requirement doesn't appear to be necessary in this case, for this would merely add two additional terms to the performance index, mainly, $q_3\ddot{\delta}_e + q_4\dot{\delta}_e$. There seems to be no apparent advantage to the inclusion of these two terms in the performance index of this problem.

Returning to the problem with the performance index of Equation 3-97, the expression for the root square locus is, as before

$$\left| I + R^{-1}G'[-Is-F']^{-1}H'[-Is-L']Q[Is-L]H[Is-F]^{-1}G \right| = 0$$

From the definition of $H[Is-F]^{-1}G$ and $G'[-Is-F']^{-1}H'$, the transfer function matrix and the transpose of the matrix with s replaced by -s, there results

$$H[Is-F]^{-1}G = \frac{\omega^2_{ACT}}{D_F(s)} \begin{bmatrix} K_\alpha(\tau_\alpha s+1) \\ K_{\dot\theta}(\tau_{\dot\theta} s+1) \end{bmatrix}$$

$$G'[-Is-F']^{-1}H' = \frac{\omega^2_{ACT}}{D_F(-s)} \begin{bmatrix} K_\alpha(-\tau_\alpha s+1) & K_{\dot\theta}(-\tau_{\dot\theta} s+1) \end{bmatrix} \tag{3-98}$$

where $K_\alpha(\tau_\alpha s+1)$ = numerator of α/δ_e (s) transfer function = N_α

$K_{\dot\theta}(\tau_{\dot\theta} s+1)$ = numerator of $\dot\theta/\delta_e$ (s) transfer function = $N_{\dot\theta}$

$K_\alpha(-\tau_\alpha s+1)$ = numerator of α/δ_e (-s) transfer function = $\bar N_\alpha$

$K_{\dot\theta}(-\tau_{\dot\theta} s+1)$ = numerator of $\dot\theta/\delta_e$ (-s) transfer function = $\bar N_{\dot\theta}$

In terms of the aerodynamic derivatives,

$$K_\alpha = M_{\delta_e} + M_{\dot\alpha} L_{\delta_e} = -2.11 \qquad K_{\dot\theta} = M_{\delta_e} L_\alpha - M_\alpha L_{\delta_e} = -1.05$$

$$\tau_\alpha = \frac{-L_{\delta_e}}{M_{\delta_e} + M_{\dot\alpha} L_{\delta_e}} = .0518 \text{ sec} \qquad \tau_{\dot\theta} = \frac{M_{\delta_e} - L_{\delta_e} M_{\dot\alpha}}{M_{\delta_e} L_\alpha - M_\alpha L_{\delta_e}} = 2.02 \text{ sec}$$

$D_F(s)$ = denominator of the transfer functions of the aircraft

$$= \underset{\text{actuator}}{(s^2 + 2\zeta\omega_{ACT} s + \omega^2_{ACT})}\ \underset{\text{aircraft}}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$D_F(-s)$ = denominator of the transfer functions of the airplane with s replaced by -s

$$= (s^2 - 2\zeta_a\omega_{ACT} s + \omega_n^2)(s^2 - 2\zeta\omega_n s + \omega_n^2)$$

Also,

$$[-Is-L'] Q [Is-L] = \begin{bmatrix} -q_1(s^2-\ell_{11}^2) + q_2 \ell_{21}^2 & q_1(s+\ell_{11}) - q_2 \ell_{21}(s-\ell_{22}) \\ -q_1(s-\ell_{11}) + q_2 \ell_{21}(s+\ell_{22}) & q_1 - q_2(s^2-\ell_{22}^2) \end{bmatrix} = \begin{bmatrix} \ell'_{11} & \ell'_{12} \\ \ell'_{21} & \ell'_{22} \end{bmatrix} \quad (3\text{-}99)$$

Substituting Equations 3-98 and 3-99 into the expression for the root square locus, there results

$$1 + \frac{(\omega_{ACT})^2}{r} \frac{[N_\alpha \bar{N}_\alpha \ell'_{11} + N_\alpha \bar{N}_{\dot\theta} \ell'_{21} + N_{\dot\theta} \bar{N}_\alpha \ell'_{12} + N_{\dot\theta} \bar{N}_{\dot\theta} \ell'_{22}]}{D_F(s)\, D_F(-s)} = 0 \quad (3\text{-}100)$$

Equation 3-100 shows that the expression for the root square locus is of the form

$$1 + \frac{KN\bar{N}}{D\bar{D}}$$

The zeros of the locus, $N\bar{N}$, are a function of q_1 and q_2 and therefore in themselves constitute a root square locus expression, which is given by

$$N_\alpha \bar{N}_\alpha \ell'_{11} + N_\alpha \bar{N}_{\dot\theta} \ell'_{21} + N_{\dot\theta} \bar{N}_\alpha \ell'_{12} + N_{\dot\theta} \bar{N}_{\dot\theta} \ell'_{22} = 0 \quad (3\text{-}101)$$

After substituting for ℓ'_{11}, ℓ'_{21}, ℓ'_{12} and ℓ'_{22} and rearranging to obtain q_1/q_2 as the parameter of the locus, there results

$$0 = 1 + \frac{q_1}{q_2} \frac{N_\alpha \bar{N}_\alpha(-s^2+\ell_{11}^2) + N_\alpha \bar{N}_{\dot\theta}(-s+\ell_{11}) + \bar{N}_\alpha N_{\dot\theta}(s+\ell_{11}) + N_{\dot\theta} \bar{N}_{\dot\theta}}{N_\alpha \bar{N}_\alpha \ell_{21}^2 + N_\alpha \bar{N}_{\dot\theta} \ell_{21}(s+\ell_{22}) + \bar{N}_\alpha N_{\dot\theta} \ell_{21}(-s+\ell_{22}) + N_{\dot\theta} \bar{N}_{\dot\theta}(-s^2+\ell_{22}^2)} \quad (3\text{-}102)$$

Equation 3-102 defines the zeros of the root square locus, that is, the roots at which the closed-loop poles of the optimal system will terminate.

After substituting for the numerical values of N_α, $\bar{N}_\alpha$, $N_{\dot\theta}$, $\bar{N}_{\dot\theta}$, ℓ_{11}, ℓ_{21}, and ℓ_{22}, Equation 3-102 becomes

$$-354 = \frac{q_1}{q_2} \frac{(s \pm 1.0285 \pm j\,4.300)}{(s \pm 2.532 \pm j\,3.237)} \quad (3\text{-}103)$$

The dotted lines of Figure 6 show half of the locus of Equation 3-102 as a function of q_1/q_2. The reflected, or adjoint, part of the locus has been omitted for clarity. As can be seen by the figure, this locus defines the end

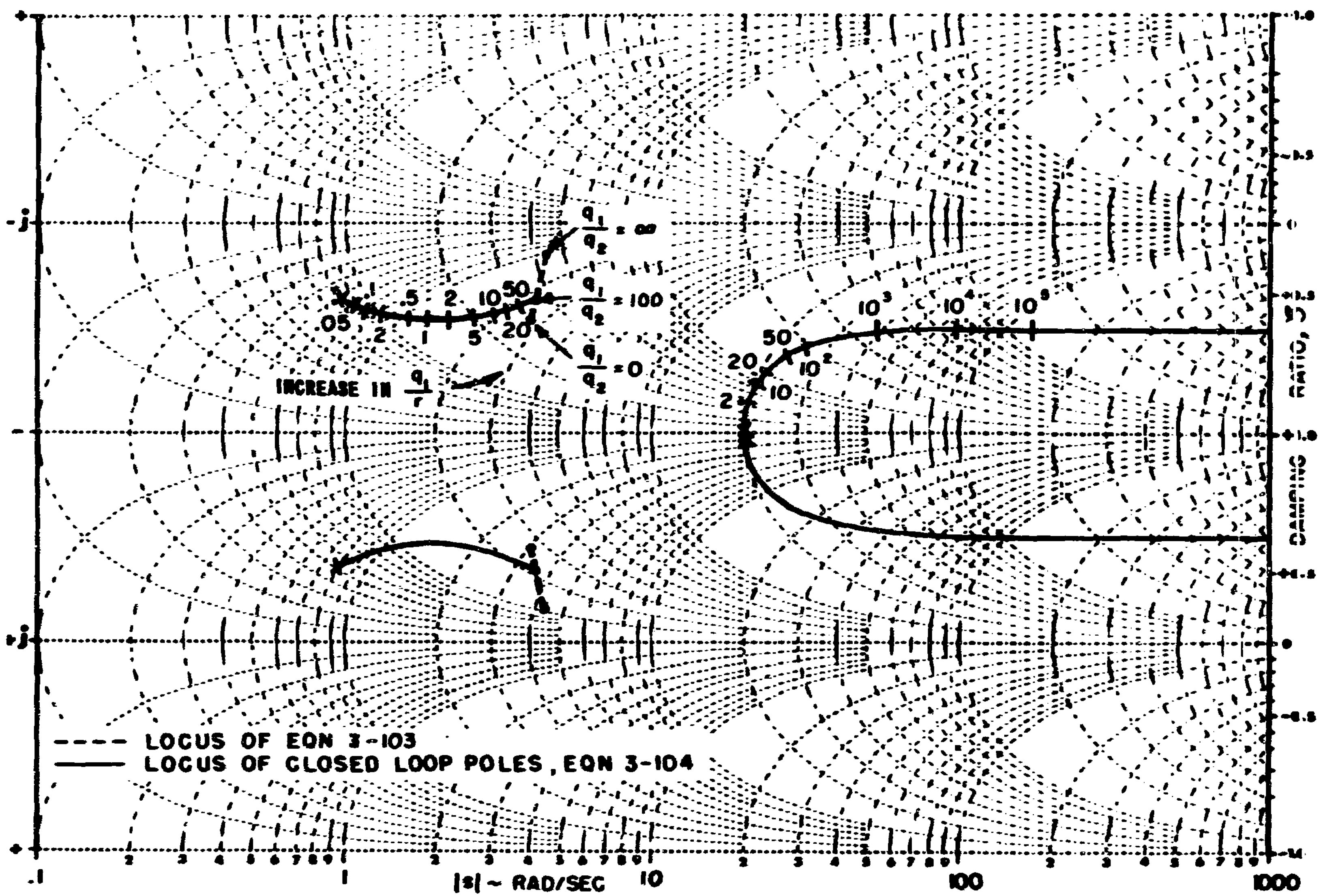


Figure 6. Locus of the Roots of the Optimal System Model in the Performance Index

points of the locus of the poles of the closed-loop optimal system. The point marked $q_1/q_2 = 0$ is the point that would have been obtained if the term $(\dot{\alpha} - \ell_{11}\alpha - \ell_{12}\dot{\theta})^2$ had been omitted from the performance index, while the point $q_1/q_2 = \infty$ is the end point that would have been obtained if the term $(\ddot{\theta} - \ell_{12}\alpha - \ell_{22}\dot{\theta})^2$ were omitted from the performance index.

The locus of the poles of the optimal system can be found once a value of q_1/q_2 has been selected. If a value of $q_1/q_2 = 100$ were chosen, Equation 3-100 can then be used to find an expression from which the closed-loop poles can be found. Substituting $q_2 = .01 q_1$ and the values for N_α, $\bar{N}_\alpha$, $N_{\dot{\theta}}$, $\bar{N}_{\dot{\theta}}$, D(s) and D(-s) into Equation 3-100 yields

$$-.000116 = \frac{q_1}{r} \; \frac{\left[s^2 \pm 2(.56)(4.19)s + (4.19)^2\right]}{\left[s \pm 20\right]^2 \left[s^2 \pm 2(.505)(.935)s + (.935)^2\right]} \qquad (3\text{-}104)$$

The locus of the roots of the optimal system, given by Equation 3-104, is shown as the solid lines in Figure 6 as a function of the ratio q_1/r. Again, the adjoint system is omitted for clarity.

The original intent, of course, was to match the system to the model as closely as possible. Because the model was second order and the aircraft was assumed to be of fourth order, it was known at the start that it would not be possible to produce an exact match. The model has a natural frequency of 4.5 rad/sec and a damping ratio of $\zeta = 0.6$. It can be seen from Figure 6 that if $q_1/q_2 = 0$, two of the closed-loop roots will in the limit have approximately the same damping ratio, $\zeta = 0.6$ as the model, but the frequency will be less than that of the model. If $q_1/q_2 = \infty$, the natural frequency of two of the closed-loop roots will approximately match the natural frequency of the model, but the damping ratio will not match. In either case, the two excess poles, originally associated with the actuator, tend toward a damping ratio of approximately 0.7.

This example attempts to answer the question "If the model roots cannot be matched exactly, how close can they be matched?" The answer, as demonstrated by the root square locus plot, is that they can be matched fairly well depending upon how the individual errors are weighted and how much control effort is allowed. If the available control effort is limited, i.e., if q_1/r is finite, the roots of the closed-loop optimal system will still systematically approach those of the model.

SECTION 4

THE SINGLE CONTROLLER LINEAR OPTIMAL SYSTEM

4.1 INTRODUCTION

R.E. Kalman stated that the major contributions of linear optimal control will be to the conceptual design of high order multi-controller multi-output control systems. It has been shown that the resulting closed-loop optimal system is stable and well behaved, with an initial condition response that always tends to approach the response of a Butterworth filter as the weighting of the error portion of the performance index is made large relative to the control. The control system designer need only select the Q and the R matrix, perhaps by trial and error, or possibly using root square locus techniques, and, using a large digital computer, an optimal control law can be computed.

A major limitation of the method just described is that the relationships that exist between the parameters of the performance index, the closed-loop dynamic characteristics, and the feedback gains are only qualitatively known. The multivariable root square locus, developed elsewhere in this report, describes in fair detail the relationships that exist between the performance index parameters and the closed-loop poles of optimal systems, but no such clear and straightforward relationships are known to exist to determine the closed form connections among the Q and R matrices of the performance index and the feedback gains of multivariable systems.

When dealing with single controller systems, however, the relationships that exist among the performance index parameters, the closed-loop optimal dynamics, and the feedback gains are not difficult to obtain. The single controller class of systems is not insignificant or unimportant and it will be instructive to demonstrate the essential relationships of single controller optimal systems.

4.2 FEEDBACK GAINS AS A FUNCTION OF Q AND R

It will be found in Section 3, Equation 3-20, that the optimal characteristic equation and adjoint are given by the determinant

$$\begin{vmatrix} Is-F & GR^{-1}G' \\ -H'QH & -Is-F' \end{vmatrix} = 0 \tag{4-1}$$

or by the spectral factored product

$$\left| Is-(F-GK) \right| \left| -Is-(F-GK)' \right| = 0 \tag{4-2}$$

Because the development here is restricted to single controller systems only, the zeros of the optimal system cannot be altered by feedback gains only. The zeros of the closed-loop system will be the same as those of the open-loop system, and only the denominator, or characteristic equation, will be altered. Equations 4-1 and 4-2 are sufficient therefore to completely describe the optimal system and its adjoint. Equating the two determinants will

therefore yield two polynomials; of which the coefficient of one is a function of Q and R, while the other is a function of the feedback gain K. Equating the coefficients will then yield a set of equations relating the feedback gains with the elements of the Q and R matrices. In addition, because the relationships between the dynamic characteristics (such as closed-loop natural frequency and damping ratio) and the feedback gains are known, these dynamic characteristics can be also related to the elements of the Q and R matrices.

Example:

Consider the plant that can be completely described by the following transfer function

$$\frac{C}{R}(s) = \frac{b}{s(s+a)} \tag{4-3}$$

whose state description can be written:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{4-4}$$

Consider a performance index for this system of the form

$$2V = \int_0^\infty (q_1 x_1^2 + q_2 x_2^2 + r u^2)\, dt \tag{4-5}$$

The characteristic equation for the optimum and adjoint is given by Equation 4-2.

$$\begin{vmatrix} Is-F & GR^{-1}G' \\ -H'QH & -Is-F' \end{vmatrix} = \begin{vmatrix} s & -1 & 0 & 0 \\ 0 & s+a & 0 & b^2/r \\ -q_1 & 0 & -s & 0 \\ 0 & -q_2 & -1 & -s+a \end{vmatrix} = 0 \tag{4-6}$$

or

$$s^4 - s^2\left(q_2 \frac{b^2}{r} + a^2\right) + q_1 \frac{b^2}{r} = 0 \tag{4-7}$$

The closed-loop optimal system matrix is of the form

$$\left[F - GK\right] = \begin{bmatrix} 0 & +1 \\ 0 & -a \end{bmatrix} - \begin{bmatrix} 0 \\ b \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & +1 \\ -bk_1 & -a-bk_2 \end{bmatrix} \tag{4-8}$$

After substituting in Equation 4-2, the characteristic equation of the optimal system becomes

$$\begin{vmatrix} s & -1 \\ +bk_1 & s+a+bk_2 \end{vmatrix} \begin{vmatrix} -s & +bk_1 \\ -1 & -s+a+bk_2 \end{vmatrix} = 0 \qquad (4\text{-}9)$$

or

$$s^4 - s^2\left[-2bk_1 + a^2 + 2abk_2 + b^2k_2^2\right] + b^2k_1^2 = 0 \qquad (4\text{-}10)$$

Equating the coefficients of powers of s of Equations 4-7 and 4-10, there results a set of equations from which k can be obtained as a function of the unknowns q_1/r and q_2/r .

$$-2k_1 + 2ak_2 + bk_2^2 = \frac{q_2 b}{r} \qquad \text{(a)}$$

$$k_1^2 = \frac{q_1}{r} \qquad \text{(b)} \qquad (4\text{-}11)$$

The polynomials, Equation 4-7 or 4-10, are of the form

$$s^4 - s^2\left(-2\omega_{n_{cl}}^2 + 4\zeta_{cl}\,\omega_{n_{cl}}\right) + \omega_{n_{cl}}^4 = \left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]\left[s^2 - 2\zeta\omega_n s + \omega_n^2\right] = 0$$

so that the solutions for k_1 and k_2 can be expressed in terms of q_1/r and q_2/r or the closed-loop natural frequency and damping. The characteristic equation of the closed-loop optimal system is, from the first determinant of Equation 4-9,

$$s^2 + s(a + bk_2) + bk_1 = s^2 + 2\zeta_{cl}\,\omega_{n_{cl}} s + \omega_{n_{cl}}^2 \qquad (4\text{-}12)$$

Because stability is guaranteed for the closed-loop optimal system, the coefficients $a+bk_2$ and bk_1 must be positive. The proper solutions of k_1 and k_2 from Equation 4-11 are

$$k_1 = +\sqrt{\frac{q_1}{r}}$$

$$k_2 = -\frac{a}{b} + \frac{a}{b}\sqrt{1 + \frac{2bk_1}{a^2} + \frac{q_2}{r}\,\frac{b^2}{a^2}} \qquad (4\text{-}13)$$

If it is desired, Equations 4-12 and 4-13 can be easily manipulated to obtain the closed-loop frequency and damping as a function of q_1/r and q_2/r .

It can be seen that direct relationships do exist between the elements of the performance index, the closed-loop characteristics, and the feedback gains. The matter of obtaining a particular optimal configuration is simply a matter of the proper selection of the performance index.

4.3 PERFORMANCE INDEX FOR RESTRICTED FEEDBACK

Using the same dynamic system of Equations 4-4 assume that it is desired to obtain the form of the performance index that yields feedback from the variable x_1 only. That is to say, position feedback only is to be allowed. What quadratic performance index will yield position feedback only? With this requirement in mind, Equation 4-9 can be reformulated, setting $k_2 = 0$.

$$\begin{vmatrix} s & -1 \\ bk_1 & s+a \end{vmatrix} \begin{vmatrix} s & -bk_1 \\ 1 & s-a \end{vmatrix} = 0 \tag{4-14}$$

or

$$(s^2 + as + bk_1)(s^2 - as + bk_1) = 0$$

$$s^4 - s^2(a^2 - 2bk_1) + b^2k_1^2 = 0 \tag{4-15}$$

Equating coefficients of Equations 4-15 and 4-7,

$$q_2 \frac{b}{r} = -2k_1 \qquad \text{(a)}$$

$$k_1^2 = \frac{q_1}{r} \qquad \text{(b)} \tag{4-16}$$

Eliminating k_1 from Equations 4-16a and Equation 4-16b, the requirement that position feedback only be allowed is

$$q_1 = \frac{b^2}{4r} q_2^2$$

or

$$q_2 = \pm \frac{2}{b}\sqrt{\frac{q_1}{r}} \tag{4-17}$$

and the resulting performance index is

$$2V = \int_0^\infty \left(q_1 x_1^2 \pm \frac{2}{b}\sqrt{\frac{q_1}{r}}\; x_2^2 + ru^2 \right) dt \tag{4-18}$$

Consider a numerical example of the limited feedback problem. Let the dynamic system be represented by the set of first-order equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \tag{4-19}$$

and let the performance index be

$$2V = \int_0^\infty \left(q_1 x_1^2 - \sqrt{q_1}\; x_2^2 + u^2 \right) dt \tag{4-20}$$

The characteristic equation of the optimal system and its adjoint is given by substituting in Equation 4-7

$$s^4 - s^2\left(4 - 4\sqrt{q_1}\right) + 4q_1 = 0 \tag{4-21}$$

The locus of the roots of the optimal system and adjoint is given in Figure 7. It can be seen that the negative sign in Equation 4-17, as selected in this problem, yields position feedback in the conventional degenerate sense. The Butterworth distribution of roots is not obtained, however. Two observations may be made from this example:

1. A Q matrix that is not positive definite or non-negative definite may still yield a stable closed-loop system.
2. It appears that the system may have closed-loop poles other than those approximating a Butterworth distribution if elements of the Q matrix are allowed to be negative.

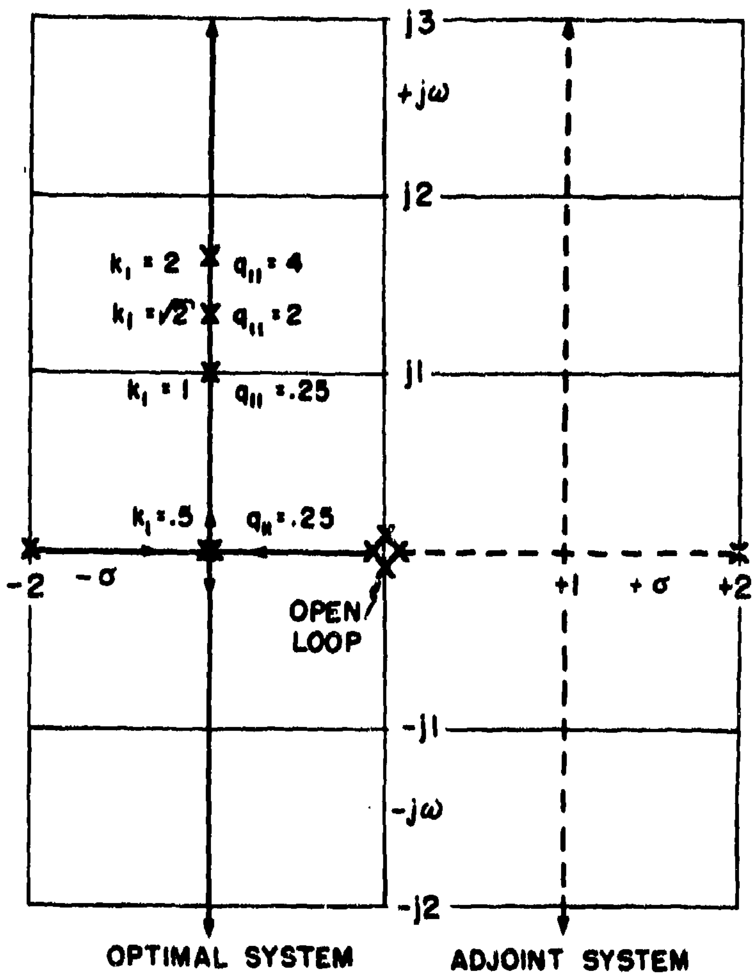


Figure 7. Locus of the Roots of the Optimal System - Position Feedback Only

Consider one other variation of the single controller problem of this section. Up to this point, the resulting closed-loop optimal system has been a regulator. No inputs other than initial conditions have been considered. Other inputs can certainly be considered, however, merely by incorporating the state variable description of this input into the plant as an uncontrollable part of the plant. Consider, for instance, a step input. This step can be described as a first-order differential equation as:

$$\dot{x}_i = 0$$

Attaching this vector to the plant of Equation 4-4, there results the system

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} x_i \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u \qquad (4\text{-}22)$$

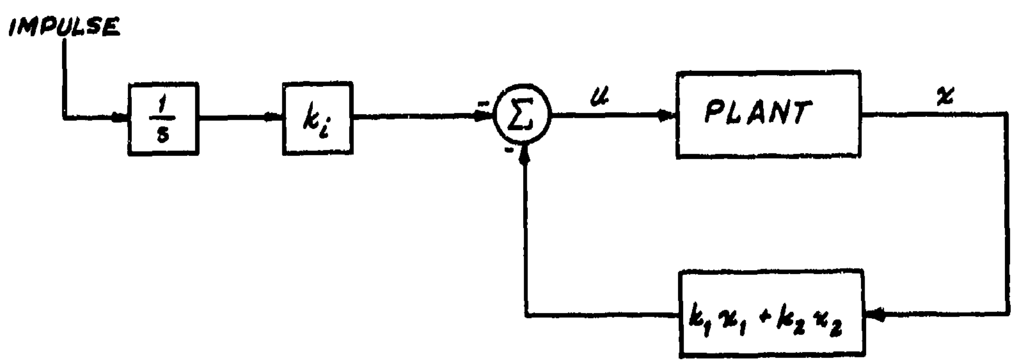
Because it can be desired to minimize the error between the step input and the output, a performance index of the following form may be selected

$$2V = \int_0^\infty \left[(x_1 - x_i)^2 q + r u^2\right] dt$$

or

$$2V = \int_0^\infty \left(q x_1^2 - 2 x_i x_1 q + x_i^2 q + r u^2\right) dt \qquad (4\text{-}23)$$

The resulting optimal system will be of the form indicated in the sketch below.



If the determinants of Equations 4-1 and 4-2 were expanded and the coefficients were equated, it would be found that k_i could not be obtained in this manner. The k_i can be found from the Riccati equation or by using S. S. L. Chang's approach. If computed, it would be found that $k_i = k_1$. It would also be found that the closed-loop part of the system is still an optimal regulator.

It has been found in general that the closed-loop regulator part of the system is not a function of the excitation to the system. The input gains to the system, such as the k_i of the above sketch, will be a function of the dynamic characteristics of the system excitation. This can be easily understood because each of the state variables required to describe the input excitation contribute to the closed-loop part of the system.

The input problem and other problems discussed in this section demonstrate that there are definite, fixed relationships that exist among the parameters of a quadratic performance index and the characteristics of the optimal closed-loop system. The simple examples have shown that much of the conventional control terminology, such as natural frequency and damping ratio, can be expressed in terms of the parameters of the quadratic performance index.

It was not the intent of this section, however, to show that everything done by optimal control could be done by conventional techniques. Optimal control yields a method whereby systems of large dimension and many inputs can be computer designed, with the knowledge that a positive definite matrix R and a non-negative definite matrix Q yield a system known to have desirable characteristics. If these two conditions are met, the system is guaranteed to be stable and the response of the closed-loop optimal system rapidly approaches the response of a Butterworth filter, smooth and well behaved.

Possibly the most important characteristic of linear optimal techniques is the freedom of the designer to qualitatively determine the amplitudes of the motions of the controller inputs to the system. If the control motions of a linear optimal design are too large, it is necessary only to increase the weighting on the controller in the performance index and the next design will have lower control motions but at the expense of greater error. In either case, the integral of the square of the control motions is minimized. This means that large excursions of the control are discouraged for any weighting of the control in the performance index. In fact, S. S. L. Chang (Reference 3) demonstrates that for the same speed of response, the linear optimal system yields lower amplitude control motions over a conventionally designed system.

SECTION 5

LONGITUDINAL SHORT PERIOD OPTIMAL FLIGHT CONTROL

5.1 INTRODUCTION

It is possible to use variational techniques to conceptually design a flight control system to satisfy flying qualities requirements for a longitudinal short period aircraft representation. By the proper choice of the flight parameters that are included in the performance index, and by proper weighting of these state and control variables, it appears that any combination of closed-loop short period natural frequency and damping may be obtained. However, conventional design techniques also exist that can do the same thing, and it is important to ask why one would choose the variational approach when conventional techniques now exist.

The answers lie in the fact that the conventional techniques do not necessarily produce a unique solution; there is more than one way to achieve the same aircraft frequency and damping. Optimal control can be used as a tool to help select a particular solution. In doing so, it has been found from experience that the control motions required to achieve a particular response are generally smoother and more well behaved. In other words, optimal control considers the behavior of the control deflections as well as the dynamic characteristics of the state variables. Because of this, it is very possible that the optimal feedback will be quite different from the feedback obtained by conventional techniques, even of a reversed sense. Therefore, although it is true that optimal control will produce nothing new for so simple an example, the principles in the use of optimal techniques will be well illustrated, and parallels in design techniques between conventional and optimal methods will become evident.

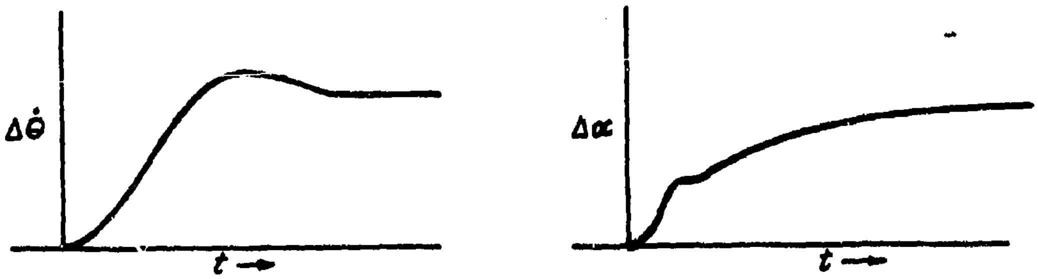
5.2 A DESIGN PHILOSOPHY

The general principles that guide a control system designer using conventional design techniques, such as root locus plots, have parallels in root square locus design. For instance, to increase the short period natural frequency, a control system engineer may specify angle of attack feedback to the elevator such that a positive incremental angle of attack change produces a positive incremental elevator deflection. Conversely, to decrease the short period frequency, a negative elevator increment is produced by a positive angle of attack change. It has been found that design by linear optimal control techniques produces similar principles of design. In order to increase the speed of response of a linear optimal system, the weighting on the state variables is made larger. To decrease the natural frequency, a weighting factor can be made negative.

To qualitatively understand how the response of a multivariable linear optimal system varies as a function of the performance index, it should be recalled that the response of the variable that appears in the performance index will approach the response of a Butterworth filter as the weighting of the output variable is made large with respect to the control variable. For instance, consider the performance index

$$2V = \int_0^{\infty} (q\,\Delta\dot{\theta}^2 + r\,\Delta\delta_e^{\,2})\,dt$$

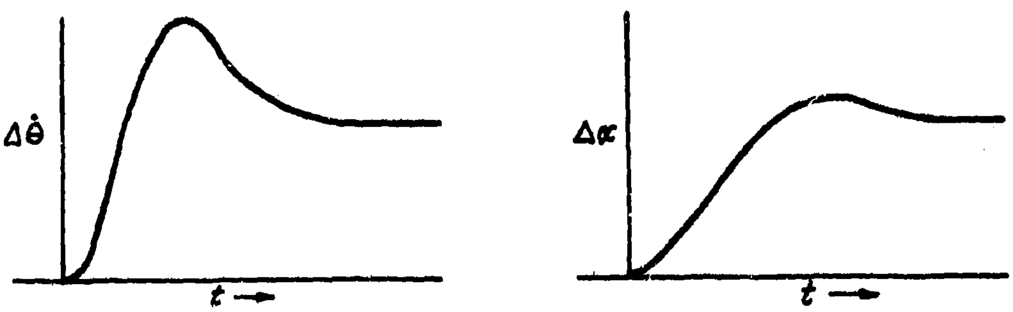
with a two-degree-of-freedom set of longitudinal equations of motion of a normal aircraft. The transient responses of the optimal system will look as sketched below:



On the other hand, an optimal design of the same aircraft, but with the performance index

$$2V = \int_0^{\infty} (q\,\Delta\alpha^2 + r\,\Delta\delta_e^{\,2})\,dt$$

will have optimal system transient responses resembling those sketched below.



Applying what is presently known about aircraft flying qualities, one can establish a clear advantage of one index over the other. In this hypothetical case, the second may be more desirable.

The normal acceleration short period response of a modern military fighter aircraft is similar to the angle of attack response with the exception that an accelerometer located at the center of gravity of the aircraft shows a delay, or a nonminimum phase characteristic. This characteristic may cause instability if Δn_z is used in a feedback design philosophy, but not if linear optimal techniques are used.

Since it can be hypothesized that an aircraft having a response approaching a Butterworth filter response in either $\Delta\alpha$ or Δn_z may be acceptable from a flying qualities point of view, it will be instructive to show how the two variables can be combined in a performance index to yield a wide range of closed-loop natural frequencies and damping ratios.

Consider the aircraft short period representation described in part by the following transfer functions

$$\frac{\Delta\alpha}{\Delta\delta_e} = \frac{K_\alpha(\tau_\alpha s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1} \qquad \frac{\Delta n_z}{\Delta\delta_e} = \frac{K_{n_z}(a_z s^2+b_z s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1} \tag{5-1}$$

where $\Delta\alpha$ = incremental angle of attack

Δn_z = incremental normal acceleration

$\Delta\delta_e$ = incremental elevator deflection

Let it be assumed that the following performance index will be used to define the optimal system

$$2V = \min_{\Delta\delta_e} \int_0^\infty \left[q(\Delta\alpha \pm k\Delta n_z)^2 + r\Delta\delta_e^2\right]dt \tag{5-2}$$

where q = weighting factor of the state variable

r = weighting factor of the control variable

k = a positive scalar

Accordingly, the transfer function to be considered for the root square locus is

$$\frac{A}{\Delta\delta_e}(s) = \frac{\Delta\alpha(s)\pm k\Delta n_z(s)}{\Delta\delta_e(s)} = \frac{K_\alpha(\tau_\alpha s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1} \pm \frac{kK_{n_z}(a_z s^2+b_z s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1} \tag{5-3}$$

The scalar k serves to alter the zeros of the over-all transfer function $A/\Delta\delta_e(s)$. It is to these altered zeros that the closed-loop roots will migrate as q/r is increased. Consider the locus of the zeros of the transfer function of Equation 5-3 with ±k as the parameter. See the sketch below:

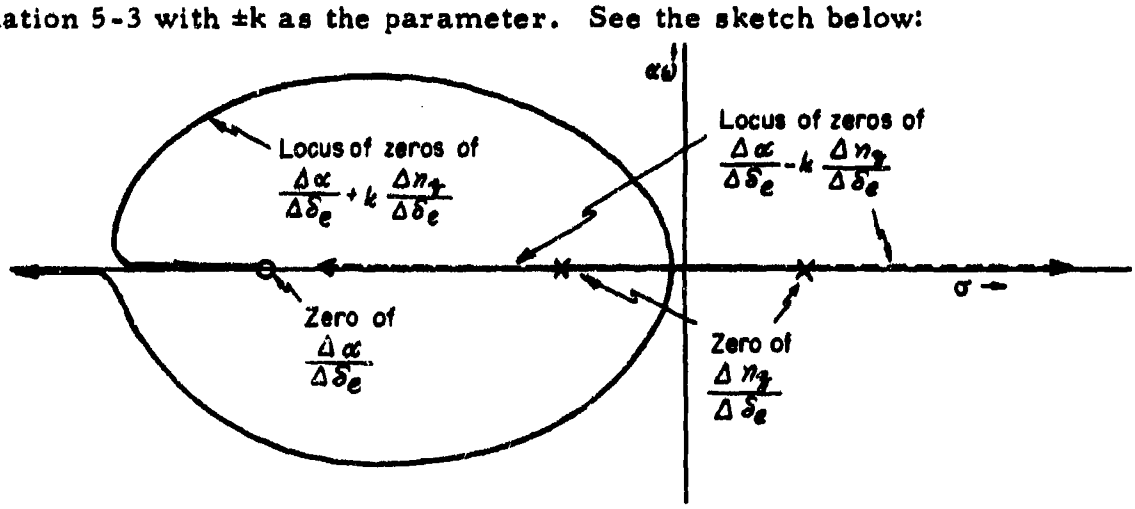
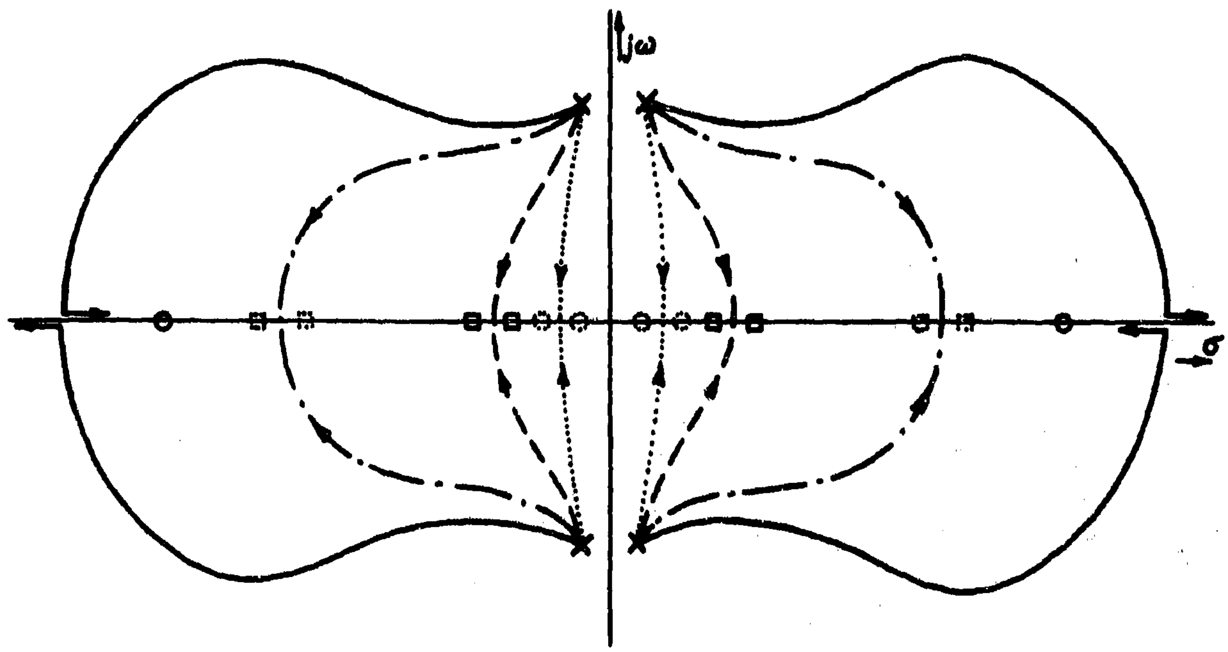


Figure 8. Locus of Zeros of $\frac{\Delta\alpha}{\Delta\delta_e}(s) \pm k\frac{\Delta n_z}{\Delta\delta_e}(s)$

The sketch on the previous page shows that the zeros of the function $\Delta\alpha/\Delta\delta_e \pm k\, \Delta n_z/\Delta\delta_e$ can be made to change almost at will along the real axis according to the sense and magnitude of k.

The importance of this to root square locus design procedures is that the poles of the real and the "adjoint" system migrate to the zeros of the real and the "adjoint" system. Since these zeros are adjustable nearly anywhere along the real axis, the poles of closed-loop optimal system can be made to migrate to nearly any point on the negative real axis.

Consider the following sketch, Figure 9, which depicts the complete root square locus for four different values of k in a performance index.



———— solid line - root square locus for $\int [q\Delta\alpha^2 + r\Delta\delta_e^{\,2}]dt$

– – – – dashed line - root square locus for $\int [q\Delta n_z^{\,2} + r\Delta\delta_e^{\,2}]dt$

—·— dot-dash line - root square locus for $\int [q(\Delta\alpha - k_1\Delta n_z)^2 + r\Delta\delta_e^{\,2}]dt$

········ dotted line - root square locus for $\int [q(\Delta\alpha + k_1\Delta n_z)^2 + r\Delta\delta_e^{\,2}]dt$

O solid circles - zeros of $\frac{\Delta\alpha}{\Delta\delta_e}(s)$ and $\frac{\Delta\alpha}{\Delta\delta_e}(-s)$

□ solid squares - zeros of $\frac{\Delta n_z}{\Delta\delta_e}(s)$ and $\frac{\Delta n_z}{\Delta\delta_e}(-s)$

⁞⁞ dotted squares - zeros of $\frac{\Delta\alpha}{\Delta\delta_e}(s) - k\frac{\Delta n_z}{\Delta\delta_e}(s), \frac{\Delta\alpha}{\Delta\delta_e}(-s) - k\frac{\Delta n_z}{\Delta\delta_e}(-s)$

◌ dotted circles - zeros of $\frac{\Delta\alpha}{\Delta\delta_e}(s) + k\frac{\Delta n_z}{\Delta\delta_e}(s), \frac{\Delta\alpha}{\Delta\delta_e}(-s) + k\frac{\Delta n_z}{\Delta\delta_e}(-s)$

Figure 9. Sketch of the Poles of the Optimal System for Several Values of k

It can be seen from the sketches that a wide range of closed-loop natural frequencies and damping ratios can be obtained simply by choosing the sense and the magnitude of the scalar k that appears in the performance index associated with each of the root square loci. By choosing +k, the natural frequency is generally increased; by choosing -k, the natural frequency is generally decreased.

Examples of this method of root square locus design have been done on the ESIAC computer. Natural frequencies and damping ratios well within the flight dynamics acceptable iso-opinion specification can be obtained.

5.3 NUMERICAL EXAMPLE OF ANALYSIS OVER A FLIGHT RANGE

An example using the equations of longitudinal motion of a modern-high-performance aircraft will serve to illustrate the manner in which the non-minimum phase characteristics can be used to good advantage, as outlined above.

The equations of motion are assumed to be given by

$$\Delta\ddot{\theta} = M_\alpha \Delta\alpha + M_{\dot\theta}\Delta\dot{\theta} + M_{\dot\alpha}\Delta\dot{\alpha} + M_\delta \Delta\delta_e \qquad \text{Pitching Acceleration Equation}$$

$$\Delta\dot{\theta} - \Delta\dot{\alpha} = L_\alpha \Delta\alpha + L_\delta \Delta\delta_e \qquad \text{Flight Path Velocity Equation}$$

$$\Delta n_z = \frac{V}{g}(\Delta\dot{\theta} - \Delta\dot{\alpha})$$

$$\tau\Delta\dot{\delta}_e + \Delta\delta_e = \Delta\delta_c \qquad \text{Actuator Dynamics}$$

Aerodynamic derivatives are hypothesized that might represent a wide range of flight conditions as indicated in Table 1 below. Transfer functions, for use in the root square locus, are next developed and tabulated in Table 2.

TABLE 1

AERODYNAMIC DERIVATIVES

Flight Condition	M_α	M_q	$M_{\dot\alpha}$	M_{δ_e}	L_α	L_{δ_e}	τ_{Acr}
A - Power Approach	-.741	-.257	-.267	-2.09	.535	.110	.05
B - M = .5 Sea Level	-4.75	-.542	-.0638	-13.1	1.09	.255	.05
C - M = .9 Sea Level	-13.0	-1.60	0	-39.4	.991	.415	.05
D - M = 1.2 Sea Level	-43.7	-2.16	0	-59.3	1.363	.438	.05
E - M = 0.6 40,000 ft	-1.87	-.194	-.0210	-4.86	.375	.0795	.05
F - M = 2 60,000 ft	-9.99	-.249	0	-9.45	.140	.0417	.05

The transfer functions are:

$$\frac{\Delta\dot{\theta}}{\Delta\delta_e}(s) = \frac{K_{\dot\theta}(\tau_{\dot\theta}s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1} \qquad \frac{\Delta\alpha}{\Delta\delta_e}(s) = \frac{K_\alpha(\tau_\alpha s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1}$$

$$\left.\frac{\Delta n_z}{\Delta\delta_e}\right)_{cg}(s) = \frac{K_{n_z}(b_{\dot z}s^2+a_{\dot z}s+1)}{\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1}$$

where:

$$K_{\dot\theta} = \frac{M_{\delta_e}L_\alpha - M_\alpha L_{\delta_e}}{-M_\alpha - M_q L_\alpha} \qquad K_\alpha = \frac{M_{\delta_e} + M_q L_{\delta_e}}{-M_\alpha - M_q L_\alpha} \qquad K_{n_z} = \frac{V}{g}K_{\dot\theta}$$

$$\tau_{\dot\theta} = \frac{M_{\delta_e} - L_{\delta_e}M_{\dot\alpha}}{M_{\delta_e}L_\alpha - M_\alpha L_{\delta_e}} \qquad \tau_\alpha = \frac{-L_{\delta_e}}{M_{\delta_e} + M_q L_\alpha}$$

$$b_{\dot z} = \frac{L_{\delta_e}}{M_{\delta_e}L_\alpha - M_\alpha L_{\delta_e}} \qquad a_{\dot z} = \frac{-L_{\delta_e}M_q - L_{\delta_e}M_{\dot\alpha}}{M_{\delta_e}L_\alpha - M_\alpha L_{\delta_e}}$$

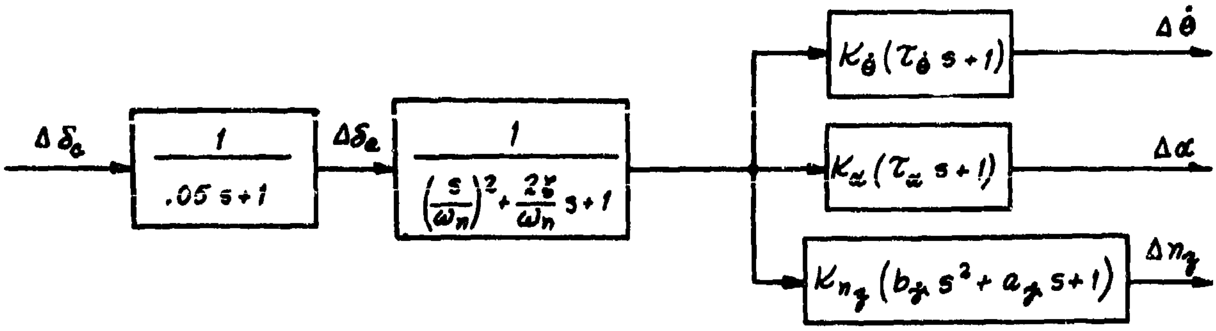
$$\omega_n = \sqrt{-M_\alpha - M_q L_\alpha} \qquad \zeta = \frac{L_\alpha - M_q - M_{\dot\alpha}}{2\sqrt{-M_\alpha - M_q L_\alpha}}$$

TABLE 2

TRANSFER FUNCTION PARAMETERS

Flt. Cond.	ω_n	ζ	$K_{\dot\theta}$	$\tau_{\dot\theta}$	K_α	τ_α	K_{n_z}	$b_{\dot z}$	$a_{\dot z}$
A	.935	.502	-1.175	-2.40	2.02	.0520	-6.90	-.106	-.030
B	2.30	.370	-2.48	-2.50	1.000	.0190	-43.0	-.0190	-.012
C	3.80	.340	-2.33	-2.75	1.16	.01035	-72.8	-.0120	-.0197
D	6.80	.260	-1.33	-1.30	0.730	.00725	-55.5	-.0071	-.0150
E	1.40	.210	-.866	-2.50	2.90	.0158	-15.7	-.0476	-.0930
F	3.70	.069	-.136	-.950	6.90	.0044	-9.08	-.0304	-.00755

A flow diagram for the two-degree-of-freedom aircraft representation is given below:



Consider the criterion of design to be given by the integral:

$$2V = \int_0^\infty \left[(\Delta\alpha + k\Delta n_z)^2 q + r\Delta\delta_e^2\right] dt \tag{5-4}$$

The root square locus expression becomes:

$$1 + \frac{q}{r}\frac{A}{\Delta\delta_c}(-s)\frac{A}{\Delta\delta_c}(s) = 0 \tag{5-5}$$

where
$$\frac{A}{\Delta\delta_c}(s) = \frac{\Delta\alpha}{\Delta\delta_c}(s) + \frac{k\Delta n_z}{\Delta\delta_c}(s) = H\left[Is-F\right]^{-1}G$$

In terms of the parameters of Table 2, $A/\delta_c\ (s)$ becomes:

$$\frac{A}{\delta_c}(s) = \frac{K_\alpha(\tau_\alpha s+1) + kK_{n_z}(b_z s^2 + a_z s+1)}{\left[\frac{s}{\tau_{ACT}}+1\right]\left[\left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n}s+1\right]} \tag{5-6}$$

It was decided to select a constant value for k and obtain the root square loci for the six flight conditions listed in Table 1. The value selected for k was +0.3. It was felt that this selection would yield a set of zeros of the root square locus plots that would in general influence the closed-loop pole positions in a desirable manner. The selection of $\Delta\alpha + k\Delta n_z$ as the state variable term of the performance index with k a constant also indicates a trade-off is made among the flight conditions as to the major contributor among the variables in the performance index. At the power approach flight condition, $K_\alpha > kK_{n_z}$ and the major contributor to the performance index is the variable $\Delta\alpha$. In a high dynamic pressure flight condition, an elevator deflection will produce normal acceleration changes that are large relative to the change in angle of attack, and the major contribution to the performance index will be made by the normal acceleration variable Δn_z. So the index of performance will reflect a desire to minimize excursions of angle of attack at the power approach and normal accelerations at high dynamic pressure flight conditions.

After substituting the numerical values for the aerodynamic derivatives into the transfer functions of Equation 5-6, the following root square locus expressions are obtained:

Power Approach:
$$-5 \times 10^{-2} = \frac{q}{r} \; \frac{\left(\pm \frac{s}{4.67}+1\right)\left(\pm \frac{s}{4.38}+1\right)}{\left(\pm \frac{s}{20}+1\right)\left[\left(\frac{s}{.935}\right)^2 \pm \frac{2(.605)}{.935}\, s+1\right]}$$

M = 0.5, Sea Level:
$$-4.2 \times 10^{-3} = \frac{q}{r} \; \frac{\left(\pm \frac{s}{7.9}+1\right)\left(\pm \frac{s}{7.8}+1\right)}{\left(\pm \frac{s}{20}+1\right)\left[\left(\frac{s}{2.31}\right)^2 \pm \frac{2(.37)}{2.31}\, s+1\right]}$$

M = 0.9, Sea Level:
$$-1.7 \times 10^{-3} = \frac{q}{r} \; \frac{\left(\pm \frac{s}{10.3}+1\right)\left(\pm \frac{s}{8.6}+1\right)}{\left(\pm \frac{s}{20}+1\right)\left[\left(\frac{s}{3.82}\right)^2 \pm \frac{2(.34)}{3.82}\, s+1\right]}$$

M = 1.2, Sea Level:
$$-3.1 \times 10^{-5} = \frac{q}{r} \; \frac{\left(\pm \frac{s}{13.4}+1\right)\left(\pm \frac{s}{11.3}+1\right)}{\left(\pm \frac{s}{20}+1\right)\left[\left(\frac{s}{6.82}\right)^2 \pm \frac{2(.26)}{6.82}\, s+1\right]}$$

M = 0.6, 40,000 ft:
$$-1.9 \times 10^{-2} = \frac{q}{r} \; \frac{\left(\pm \frac{s}{4.9}+1\right)\left(\pm \frac{s}{6.7}+1\right)}{\left(\pm \frac{s}{20}+1\right)\left[\left(\frac{s}{1.39}\right)^2 \pm \frac{2(.21)}{1.39}\, s+1\right]}$$

M = 2.0, 60,000 ft:
$$-7.4 \times 10^{-2} = \frac{q}{r} \; \frac{\left(\pm \frac{s}{6.6}+1\right)\left(\pm \frac{s}{6.8}+1\right)}{\left(\pm \frac{s}{20}+1\right)\left[\left(\frac{s}{3.7}\right)^2 \pm \frac{2(.069)}{3.7}\, s+1\right]}$$

The actual root square locus plots for the optimal systems with q/r as the parameter are shown in Figures 10 through 15. For clarity, the locus of the adjoint system was not included in the plots.

It can be seen that the poles of the closed-loop optimal system originally associated with the short period mode of motion become more highly damped and have a higher natural frequency. In general, flying qualities requirements indicate that this is the proper adjustment to make for this particular aircraft. There is one noticeable exception, however. The closed-loop frequency for the flight condition of Mach = 1.2, Sea Level, would probably be too high for any value of q/r. This comes about because the original open-loop aircraft appears to have a sufficiently rapid response without augmentation.

The actual implementation of the type of system conceptually indicated by the root square locus plots would require a programmed feedback gain schedule as a function of flight condition. It would be necessary to tabulate the gains to determine the complexity and soundness of such a program, and to obtain simplifying approximations to the exact optimal control law. It is nevertheless demonstrated that if a gain program were feasible, an optimal approach to flight control system design is possible, and the object of the preceding exercise has been achieved.

The root square locus analysis performed in this section is really one of the first steps to be performed in a complete analysis of a flight control system. Additional characteristics such as feedback gains, transient responses and sensor dynamics and location must be investigated. Experimentation must be undertaken to determine pilot reaction to the new closed-loop characteristics of the stability augmented vehicle. In short, many factors must be investigated in addition to closed-loop root locations, but the closed-loop roots are important and they can be easily controlled using linear optimal control techniques.

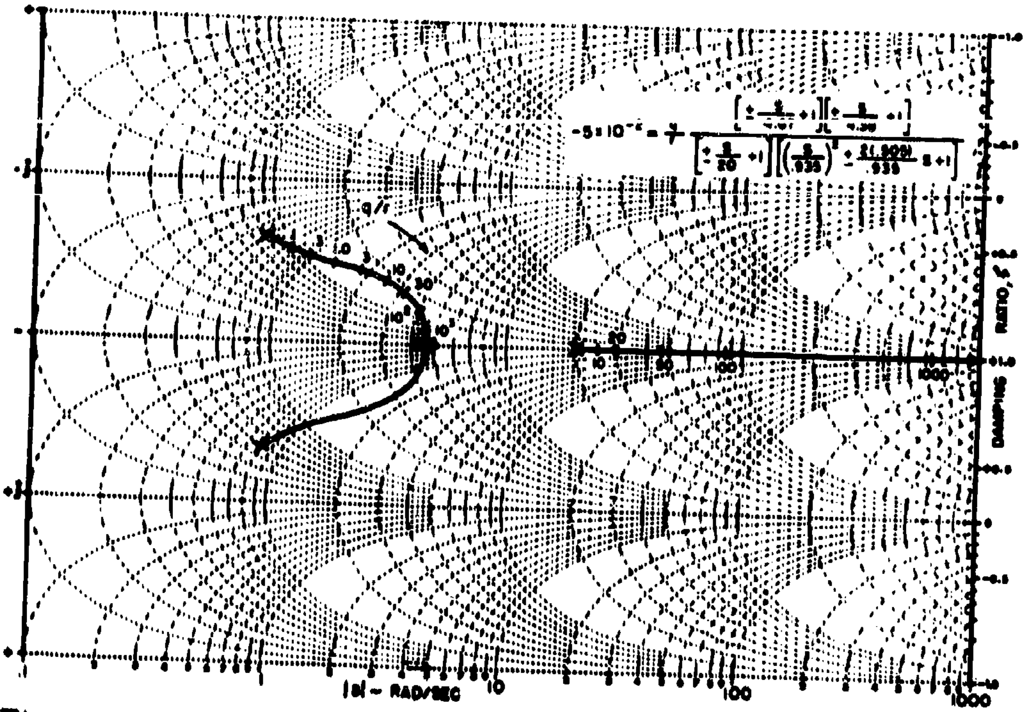


Figure 10. Root Square Locus for the Performance Criterion $2V = \min_{\Delta\delta_e} \int_0^\infty \left[(\Delta\alpha + .3\Delta n_z)^2 q + r\Delta\delta_e^2\right] dt$ ~ Power Approach

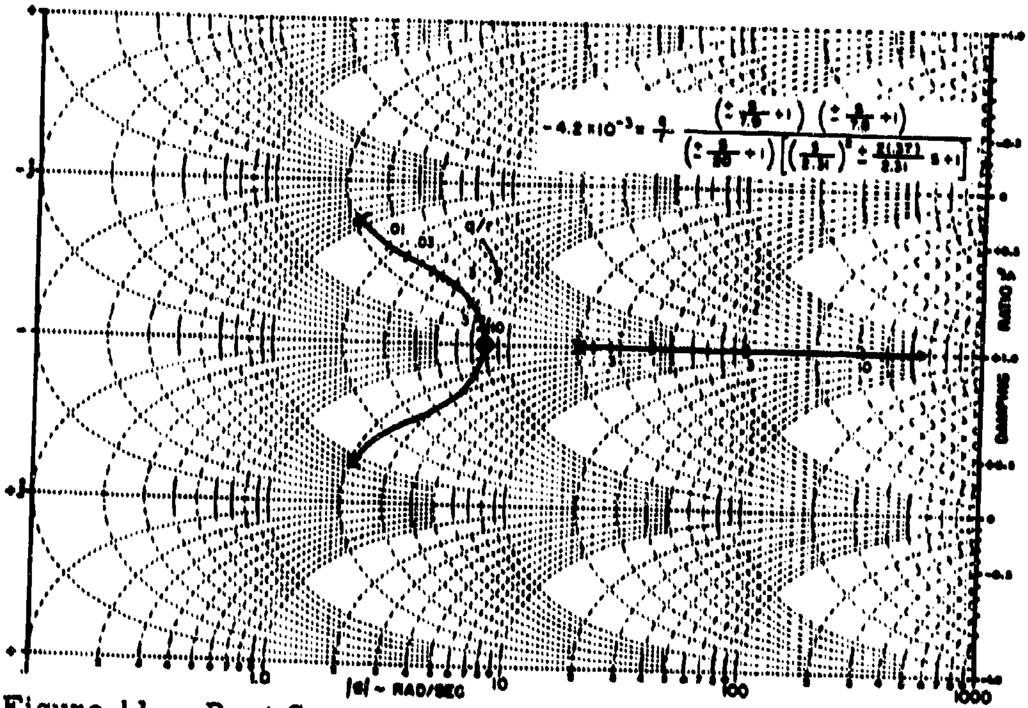


Figure 11. Root Square Locus for the Performance Criterion $2V = \min_{\Delta\delta_e} \int_0^\infty \left[(\Delta\alpha + .3\Delta n_z)^2 q + r\Delta\delta_e^2\right] dt$ ~ Mach = 0.5, h = Sea Level

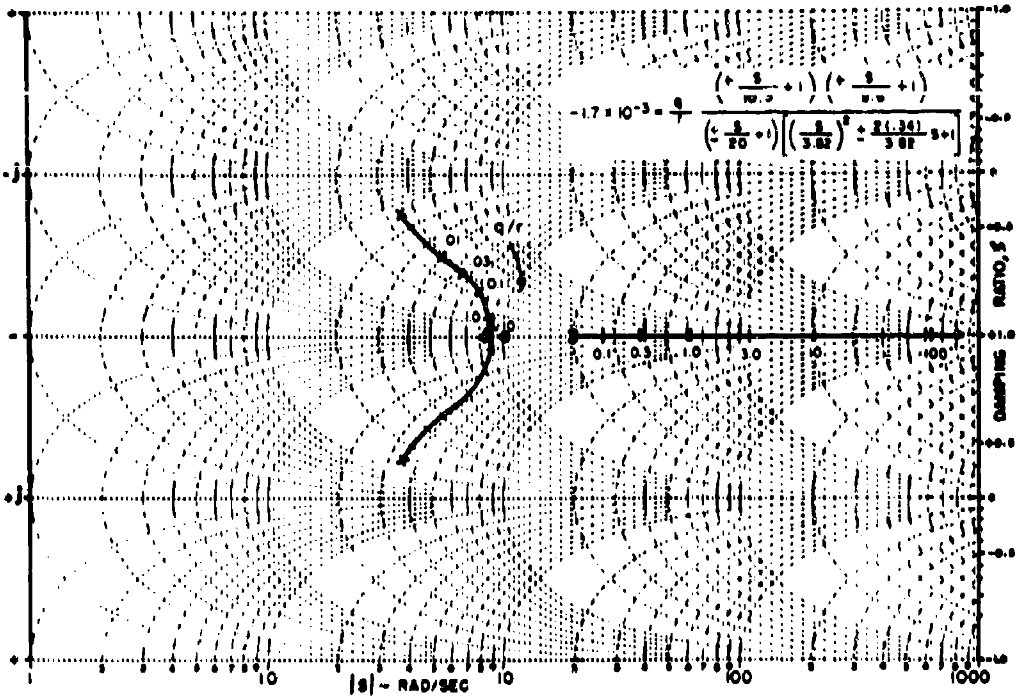


Figure 12. Root Square Locus for the Performance Criterion
$2V = \min_{\Delta\delta_e} \int_0^\infty \left[(\Delta\alpha + .3\Delta n_z)^2 q + r\Delta\delta_e^2 \right] dt$ ~ Mach = 0.9, h = Sea Level

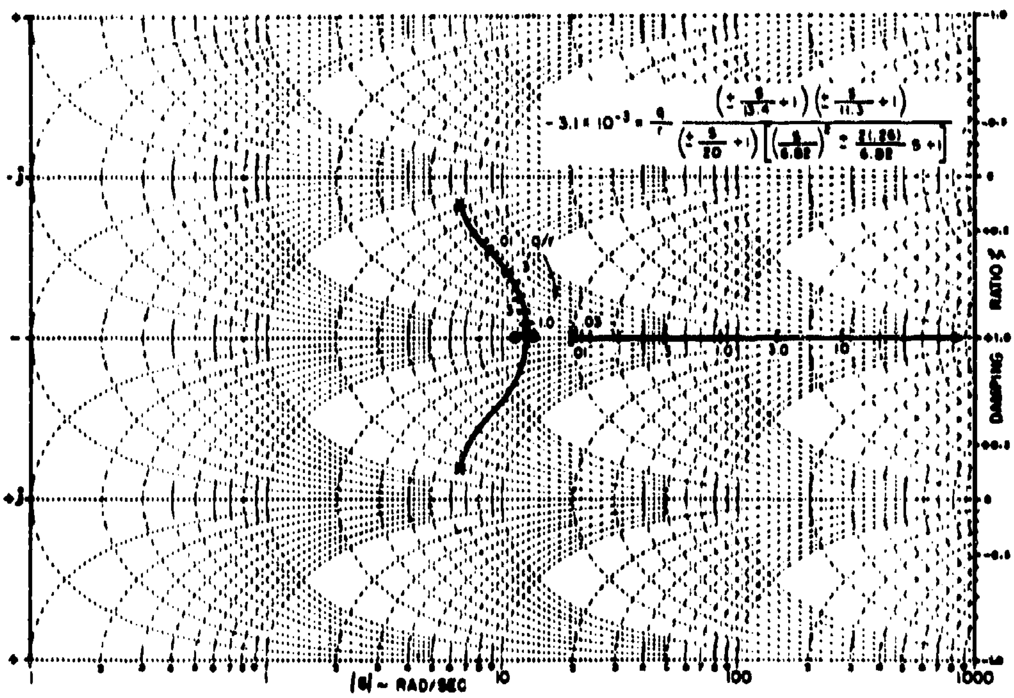


Figure 13. Root Square Locus for the Performance Criterion
$2V = \min_{\Delta\delta_e} \int_0^\infty \left[(\Delta\alpha + .3\Delta n_z)^2 q + r\Delta\delta_e^2 \right] dt$ ~ Mach = 1+, h = Sea Level

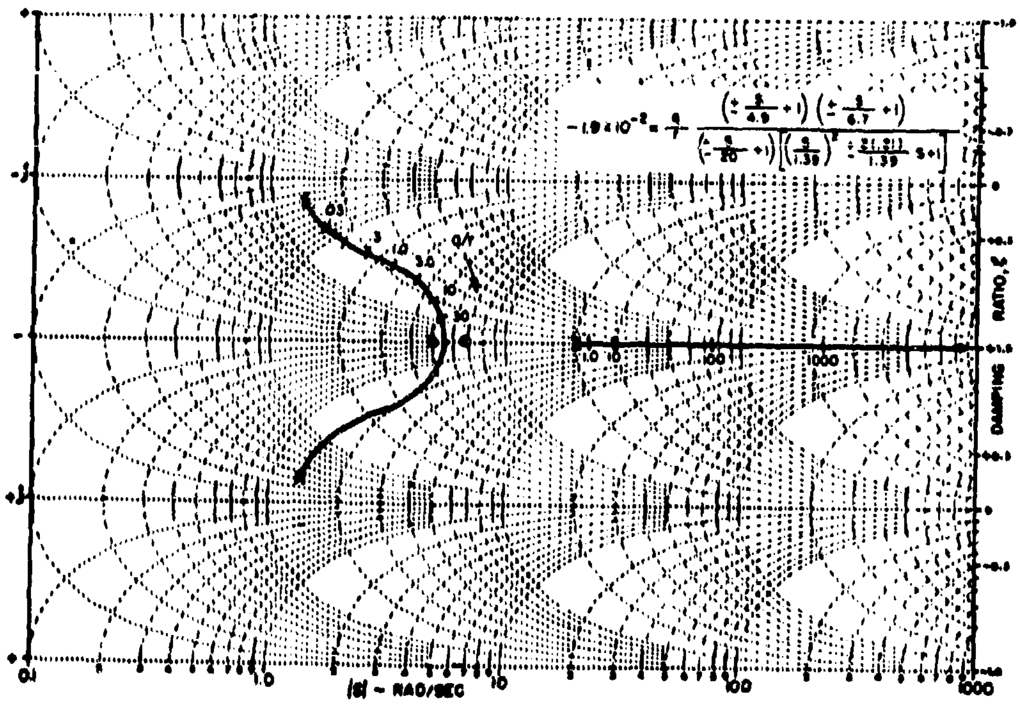


Figure 14. Root Square Locus for the Performance Criterion $2V = \min_{\Delta\delta_e} \int_0^\infty \left[(\Delta\alpha + .3\Delta n_z)^2 q + r\Delta\delta_e^2\right] dt$ ~ Mach = 0.6, h = 40,000 ft

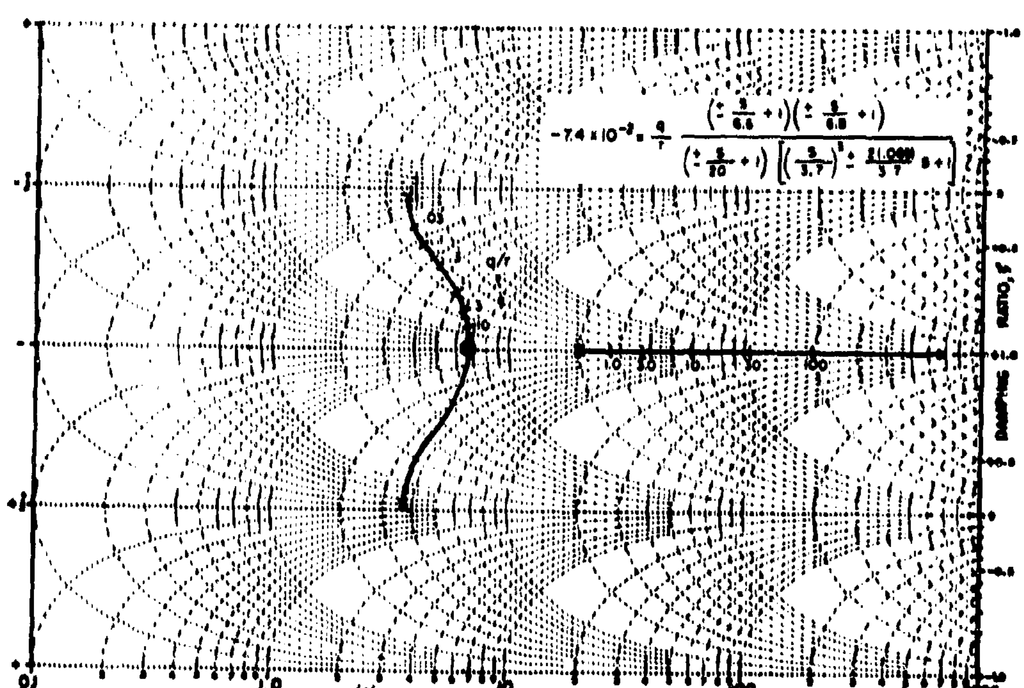


Figure 15. Root Square Locus for the Performance Criterion $2V = \min_{\Delta\delta_e} \int_0^\infty \left[(\Delta\alpha + .3\Delta n_z)^2 q + r\Delta\delta_e^2\right] dt$ ~ Mach = 2+, h = 60,000 ft

SECTION 6
MINIMIZATION IN THE FREQUENCY DOMAIN – THE SINGLE VARIABLE PROBLEM

6.1 INTRODUCTION

For the most part, the previous sections have emphasized a time domain approach to linear optimal control which relied on the conventional techniques of the calculus of variations for its theoretical development. However, there is merit in covering essentially the same material using a frequency domain approach, if for no other reason than that there are a large number of control engineers who would prefer it this way.

In this section, and the two which follow it, a frequency domain approach to linear optimal control is developed which requires the use of analysis methods which, for the most part, are familiar to the majority of engineers (e.g., root locus, solution of linear sets of equations using Cramer's Rule, etc.).

In this section, a brief survey of the technique for minimizing a quadratic performance index with only one available control variable, in the manner developed by S.S.L. Chang, is presented. For this case, it is shown that the frequency domain equation of interest is a scalar equation of the Wiener-Hopf type that can be solved using what is now considered to be a standard technique (spectral factorization). The theory is developed for both the optimal transfer function and the optimal control with examples given to demonstrate the application of the latter. The section concludes by giving one possible solution to the "problem of the type zero plant" and for this problem demonstrates equivalence between the time domain approach and frequency domain approach.

The extension of the methods of this section to the multi-control case is given in Section 7.

6.2 THEORY FOR THE OPTIMUM TRANSFER FUNCTION

In Figure 16, W(s) represents the fixed elements of the system. The compensating network, W_c, is to be designed in such a fashion that the expression

$$\int_0^\infty e(t)^2\,dt + k^2\int_0^\infty u(t)^2\,dt \qquad (6\text{-}1)$$

is minimized.

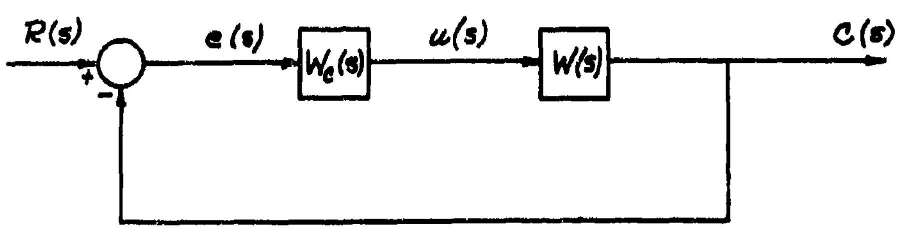


Figure 16. Single Control Optimal System

In Equation 6-1 $J_1 = \int_0^\infty e^2\,dt$ is the integral square error and $J_2 = \int_0^\infty u^2\,dt$ represents the integral square value of the control effort.

Chang's theory is developed by obtaining an equivalent frequency domain expression for Equation 6-1 in terms of the system transfer function through the use of Parseval's Theorem. The word equivalent is emphasized, for one must continually bear in mind the conditions under which the time domain expression is equal to the frequency domain expression.

Before proceeding, a basic characteristic of Parseval's Theorem should be pointed out. The theorem states

$$\int_0^\infty a(t)^2\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} A(s)\,A(-s)\,ds \qquad (6\text{-}2)$$

where

$$\mathcal{L}\left[a(t)\right] = \int_0^\infty a(t)\,e^{-st}\,dt = A(s) \qquad (6\text{-}3)$$

Suppose A(s) has poles and zeros in the right-half plane. Under these conditions, the left-hand side of Equation 6-2 does not exist. Consider now the right-hand side of Equation 6-2. The evaluation of an integral expression such as this is easily carried out through the use of standard tables (Reference 11).

It is always possible to write

$$A(s) = A_1(s)\,A_2(s) \qquad (6\text{-}4)$$

where $A_1(s)$ contains only left-half plane poles and zeros and $A_2(s)$ is the all-pass network necessary to maintain the equality of Equation 6-4. An example will make this clearer. Suppose

$$A(s) = \frac{-\tau_1 s + 1}{(-\tau_2 s + 1)(-\tau_3 s + 1)} \qquad (6\text{-}5)$$

Equation 6-5 can then be written:

$$A(s) = \underbrace{\frac{\tau_1 s + 1}{(\tau_2 s + 1)(\tau_3 s + 1)}}_{A_1(s)} \cdot \underbrace{\left(\frac{-\tau_1 s + 1}{\tau_1 s + 1}\right)\left(\frac{\tau_2 s + 1}{-\tau_2 s + 1}\right)\left(\frac{\tau_3 s + 1}{-\tau_3 s + 1}\right)}_{A_2(s)} \qquad (6\text{-}6)$$

Consider now

$$\begin{aligned} A(s)A(-s) &= A_1(s)\,A_1(-s)\,A_2(s)\,A_2(-s) \\ &= A_1(s)A_1(-s) \end{aligned} \qquad (6\text{-}7)$$

This is true because the product of an all-pass network with its conjugate is unity, that is,

$$A_2(s)\,A_2(-s) = 1 \qquad (6\text{-}8)$$

This, of course, means that the expression

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} A(s)A(-s)\,ds$$

is evaluated as though A(s) had only left-half plane poles and zeros. It is all too easy to forget, when one works strictly with the frequency domain expressions, that the equivalence between the two sides of the equations is voided

when A(s) and A(-s) no longer possess a common convergence factor. That is, there is no longer an analytic continuation from the left-half to the right-half plane, and vice versa.

We return now to Equation 6-1 and apply Parseval's theorem:

$$\int_0^\infty [e^2(t)+k^2u^2]dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} e(s)e(-s)ds + \frac{k^2}{2\pi j}\int_{-j\infty}^{j\infty} u(s)u(-s)ds \tag{6-9}$$

Equation 6-9 can be rewritten in terms of the transfer function

$$\frac{\mathcal{C}}{\mathcal{R}} = F(s)^{\dagger} \tag{6-10}$$

and becomes, after a few manipulations,

$$J_1 + k_2 J_2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [F(s)-1][F(-s)-1]\,\mathcal{R}(s)\mathcal{R}(-s)ds + \frac{k^2}{2\pi j}\int_{-j\infty}^{j\infty} \frac{F(s)F(-s)\mathcal{R}(s)\mathcal{R}(-s)}{W(s)W(-s)}ds \tag{6-11}$$

It is now required that one find the F(s) that minimizes Equation 6-11, that is,

$$J_1 + k^2 J_2 = min. \tag{6-12}$$

An arbitrary F(s) is written as

$$F(s) = F_0(s) + \lambda F_1(s)$$

where λ is a constant, F_0 is the optimum transfer function, and F_1 is any arbitrary, but physically realizable, transfer function. In other words, one takes a variation on F(s). The result is

$$J_1 + k^2 J_2 = J_a + \lambda(J_b + J_c) + \lambda^2 J_d \tag{6-13}$$

where

$$J_a = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[(F_0-1)(\bar{F}_0-1) + \frac{k^2F_0\bar{F}_0}{W\bar{W}}\right]\mathcal{R}\bar{\mathcal{R}}\,ds$$

$$J_b = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[(\bar{F}_0-1) + \frac{k^2\bar{F}_0}{W\bar{W}}\right]\mathcal{R}\bar{\mathcal{R}}F_1\,ds$$

---

†Not to be confused with the time domain F matrix used elsewhere in this report.

$$J_c = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[(F_o - 1) + \frac{k^2 F_o}{W\overline{W}} \right] R \overline{R}\, \overline{F}_1 \, ds$$

$$J_d = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[1 + \frac{k^2}{W\overline{W}} \right] F_1 \overline{F}_1 \, R \overline{R} \, ds$$

and $\overline{F} = F(-s)$, *etc.*

J_a is the "optimum", J_d is > 0 and $J_b = J_c$, since $J_b\,(s) = J_c(-s)$. From this it follows that a necessary and sufficient condition for F_o to give the lowest value of $J_1 + k^2 J_2$ is, for arbitrary $\overline{F}_1$, $J_c = 0$ (see, for example, Reference 3).

Thus we are led to consider the equation

$$\int_{-j\infty}^{j\infty} \left[(F_o - 1) + \frac{k^2 F_o}{W\overline{W}} \right] R \overline{R}\, \overline{F}_1 \, ds = 0 \tag{6-14}$$

where all the poles of $\overline{F}_1$ are inside the right-half plane. Equation 6-14 is identically zero if all the poles of

$$\left[(F_o - 1) + \frac{k^2 F_o}{W\overline{W}} \right] R \overline{R} \tag{6-15}$$

are also inside the right-half plane, and the path of integration is completed to the left.

The behavior of the integrand on the semicircular contour must be examined to assure that it vanishes. Thus one must be careful that Equation 6-15 behaves like $1/s^2$ as $s \to \infty$, assuring the convergence of the integral on the semicircular contour. This condition is usually overlooked and can be easily violated (for example, when $R(s) = 1$, it can occur that $(F_o - 1) + k^2 F_o / W\overline{W}$ is not a proper rational fraction in s).

At any rate, it follows that a necessary and sufficient condition for a minimum is that

$$\left[(F_o - 1) + \frac{k^2 F_o}{W\overline{W}} \right] R \overline{R} = q(s) \tag{6-16}$$

have right-half plane poles.

Equation 6-16 is then rewritten as

$$\left(1 + \frac{k^2}{W\overline{W}} \right) R \overline{R} F_o - R \overline{R} = q(s) \tag{6-17}$$

Let

$$1 + \frac{k^2}{W\bar{W}} = Y(s)\,Y(-s) \tag{6-18}$$

and note that Equation 6-18 is not changed by replacing s by -s.

Now, Chang defines

$$Y(s) = \left\{1 + \frac{k^2}{W\bar{W}}\right\}^+ \tag{6-19}$$

to be a function with all left-half plane poles and zeros. The need for the poles to be in the left-half plane cannot be justified. However, the need for left-half plane zeros will become apparent in Equation 6-21. The process for finding Y(s) is called spectral factorization. Chang then defines a function

$$\mathcal{Z}(s) = \left\{\mathcal{R}(s)\,\mathcal{R}(-s)\right\}^+$$

in a similar manner. Using these expressions, Equation 6-17 becomes

$$Y\bar{Y}\mathcal{Z}\bar{\mathcal{Z}}F_0 - \mathcal{Z}\bar{\mathcal{Z}} = \mathfrak{z}(s)$$

or

$$Y\mathcal{Z}F_0 - \frac{\bar{\mathcal{Z}}}{\bar{Y}} = \frac{\mathfrak{z}(s)}{\bar{Y}\bar{\mathcal{Z}}} \tag{6-20}$$

$\bar{\mathcal{Z}}/\bar{Y}$ is now expanded in partial fractions to give

$$\frac{\bar{\mathcal{Z}}}{\bar{Y}} = \left[\frac{\bar{\mathcal{Z}}}{\bar{Y}}\right]_+ + \left[\frac{\bar{\mathcal{Z}}}{\bar{Y}}\right]_-$$

where $[\bar{\mathcal{Z}}/\bar{Y}]_+$ contains the terms with left-half plane poles and $[\bar{\mathcal{Z}}/\bar{Y}]_-$ contains the terms with right-half plane poles. Equation 6-20 becomes

$$Y\mathcal{Z}F_0 - \left[\frac{\bar{\mathcal{Z}}}{\bar{Y}}\right]_+ = \frac{\mathfrak{z}}{\bar{Y}\bar{\mathcal{Z}}} + \left[\frac{\bar{\mathcal{Z}}}{\bar{Y}}\right]_- \tag{6-21}$$

Since all the poles of the expression on the left-hand side of Equation 6-21 are left-half plane and all the poles on the right side are right-half plane, the optimum solution is (see Reference 3):

$$F_0 = \frac{1}{Y\mathcal{Z}}\left[\frac{\bar{\mathcal{Z}}}{\bar{Y}}\right]_+ \tag{6-22}$$

From Equation 6-11, we see that J_2 becomes unbounded when W(s) has right-half plane zeros and F(s) does not have the same zeros as W(s). Similarly, if R(s) is $1/s^n$, then J_1 becomes unbounded if [F(s) - 1] does not have a free s^n term. Also, J_2 becomes unbounded when $\mathcal{R} = 1/s^n$ and W(s) does not have a free $1/s^n$.

One of the difficult points in the theory can be remedied immediately. It is the requirement in Equation 6-19 that Y(s) be a function with left-half plane poles which causes J_2 to be nonexistent when W(s) is non-minimum

phase. It is apparent that this difficulty can be remedied by letting F_o have the open-loop zeros. That this is the correct thing to do can be proven by going back and solving for the optimal control u_o rather than the optimal transfer function F_o. This will be done in the next section, since the expressions for the optimal controls will prove to be the real quantities of interest in the multidimensional case.

6.3 THE OPTIMUM CONTROL

In this section, S. S. L. Chang's method for solving the least-square optimization problem is reformulated by solving for the control which minimizes the performance index (rather than the transfer function which minimizes the index). This leads one directly to an expression for the system transfer function which correctly indicates that the closed-loop zeros are the open-loop zeros.

Single-Input, Single-Output

Consider the block diagram of Figure 17 (the same as Figure 16):

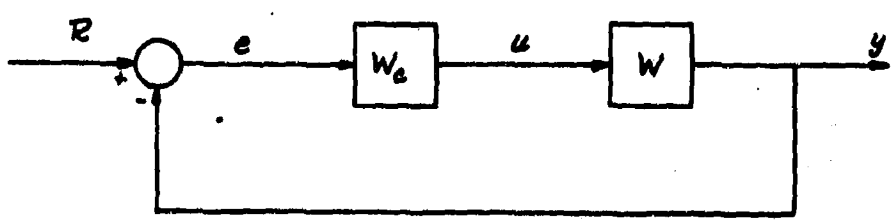


Figure 17. Single Control - Single Output Optimal System

In Figure 17, W represents the fixed elements of the system. The performance index

$$V = \int_0^\infty (e^2 + k^2 u^2)\,dt \tag{6-23}$$

is to be minimized by solving for the optimal control, u_o.

Equation 6-23 can be rewritten in the equivalent frequency domain form

$$V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left\{ (R - Wu)(\bar{R} - \bar{W}\bar{u}) + k^2 u\bar{u} \right\} ds \tag{6-24}$$

since

$$y = Wu \tag{6-25}$$

One now takes a variation on Equation 6-24 by letting

$$u = u_o + \lambda u_1$$

where u_1 is physically realisable, but otherwise arbitrary. The result is

$$2V = J_a + \lambda (J_b + J_c) + \lambda^2 J_d \tag{6-26}$$

where J_a is the optimum, J_d is > 0 and $J_b = J_c$, since $J_b(s) = J_c(-s)$.

Thus the necessary and sufficient condition for u_0 to give the lowest value of V is, for arbitrary u_1, $J_a = 0$. We must therefore consider the equation

$$J_a = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [k^2 u_0 - \bar{W}(R - W u_0)] \bar{u}_1 \, ds \tag{6-27}$$

From Equation 6-27, it is apparent that

$$(k^2 + W\bar{W}) u_0 - \bar{W} R = z(s) \tag{6-28}$$

must have right-half plane poles.

Let $$k^2 + W\bar{W} = Y\bar{Y} = \frac{N\bar{N} + k^2 D\bar{D}}{D\bar{D}}$$

and let

$$Y = \frac{\{N\bar{N} + k^2 D\bar{D}\}^+}{D} \tag{6-29}$$

where $W = N/D$

Here we let Y have as poles all of the open-loop poles, D, in order to insure that Equation 6-24 will exist when there are open-loop poles in the right-half plane. Equation 6-28 becomes

$$Y\bar{Y} u_0 - \bar{W} R = z(s) \tag{6-30}$$

and the optimum value of u is

$$u_0 = \frac{1}{Y} \left[\frac{\bar{W} R}{\bar{Y}} \right]_+ = \frac{D}{\{N\bar{N} + k^2 D\bar{D}\}^+} \left[\frac{\bar{N} R}{\{N\bar{N} + k^2 D\bar{D}\}^-} \right]_+ \tag{6-31}$$

Hence

$$F_0 = \frac{y}{R} = \frac{W u_0}{R} = \frac{W}{YR} \left[\frac{\bar{W} R}{\bar{Y}} \right]_+ \tag{6-32}$$

or

$$F_0 = \frac{N}{R\{N\bar{N} + k^2 D\bar{D}\}^+} \left[\frac{\bar{N} R}{\{N\bar{N} + k^2 D\bar{D}\}^-} \right]_+ \tag{6-33}$$

In the above expressions, it has been assumed (for convenience) that R has left-half plane poles and zeros.

Equation 6-33 is the desired result since it shows clearly that the optimum transfer function must retain the original zeros of the system.

The procedure outlined above also demonstrates the feasibility of minimizing the integral expressions by solving for the optimum control expressions rather than for the optimum transfer function.

In the sections which follow, several examples will be be given to demonstrate the application of Equation 6-31 and, more specifically, how one spectral factorizes the expression

$$k^2 + W\bar{W} \tag{6-34}$$

It is interesting to note that one usually tries to find only the optimal transfer function or optimal control expression in the single variable case. The compensating network required to force a particular optimal control is usually ignored. Apparently the feeling is that the compensation can be realized in a variety of ways. While this may be true in the single variable case, one is forced to admit that the more complex multivariable case is something else again. One of the most important tasks of the multivariable theory is to specify the "feedback gains" which compensate the open-loop systems.

6.4 EXAMPLES

For the first example one supposes, in Figure 17, that

$$W(s) = \frac{K(as+1)}{s(\tau s+1)} \tag{6-35}$$

and

$$R = \frac{1}{s}, \tag{6-36}$$

a step input.

Substituting into Equation 6-29,

$$\frac{\{N\bar{N} + k^2 D\bar{D}\}^+}{D} = Y$$

gives

$$\frac{\{K^2(-a^2s^2+1) - k^2s^2(-\tau^2s^2+1)\}^+}{s(\tau s+1)} = Y \tag{6-37}$$

Therefore

$$Y = \frac{\{k^2\tau^2s^4 - (a^2K^2+k^2)s^2 + K^2\}^+}{s(\tau s+1)} \tag{6-38}$$

To spectral factorize the numerator of Equation 6-38, one observes

$$(C_1s^2 + C_2s + C_3)(C_1s^2 - C_2s + C_3) = C_1^2s^4 - (C_2^2 - 2C_1C_3)s^2 + C_3^2 \tag{6-39}$$

Equating coefficients between Equations 6-38 and 6-39 gives

$$\begin{aligned} C_1 &= k\tau \\ C_3 &= K \\ C_2 &= \sqrt{a^2K^2 + k^2 + 2k\tau K} \end{aligned} \tag{6-40}$$

Equation 6-38 reduces to

$$Y = \frac{\left\{(C_1 s^2 + C_2 s + C_3)(C_1 s^2 - C_2 s + C_3)\right\}^+}{s(\tau s + 1)} = \frac{C_1 s^2 + C_2 s + C_3}{s(\tau s + 1)} \tag{6-41}$$

The expression for the optimal control, Equation 6-31, becomes

$$u_0 = \frac{s(\tau s + 1)}{(C_1 s^2 + C_2 s + C_3)} \left[\frac{K(-as+1)}{s(C_1 s^2 - C_2 s + C_3)} \right]_+ \tag{6-42}$$

In Equation 6-42, the only pole which contributes to the partial fraction expansion is the one due to the step input, since the poles of $C_1 s^2 - C_2 s + C_3$ are in the right-half plane.

Therefore:

$$u_0 = \frac{s(\tau s + 1)}{C_1 s^2 + C_2 s + C_3} \; \frac{\left[\dfrac{K(-as+1)}{C_1 s^2 - C_2 s + C_3}\right]_{s=0}}{s}$$

$$= \frac{(\tau s + 1)}{C_1 s^2 + C_2 s + C_3} \tag{6-43}$$

The optimal transfer function is then

$$F_0 = \frac{1}{R} u_0 W = s \frac{\tau s + 1}{C_1 s^2 + C_2 s + C_3} \cdot \frac{K(as+1)}{s(\tau s + 1)}$$
$$= \frac{K(as+1)}{C_1 s^2 + C_2 s + C_3} \tag{6-44}$$

It is easy to see that the closed-loop poles migrate towards

$$s = -\infty, \frac{1}{a} \tag{6-45}$$

as the error is weighted more and more heavily (i.e., as $k^2 \rightarrow 0$).

From Figure 17, the expression for the compensating network is found to be

$$W_C = \frac{F_0}{W(1-F_0)} \tag{6-46}$$

Substituting in the known values gives

$$W_C = \frac{\tau s + 1}{C_1 s + (C_2 - Ka)} = \frac{\left(\dfrac{1}{C_2 - Ka}\right)(\tau s + 1)}{\left(\dfrac{C_1}{C_2 - Ka}\right) s + 1} \tag{6-47}$$

We first observe that W_c is not a pr per rational polynomial. As $k^2 \to 0$ (weighting error term heavily)

$$W_c = \frac{\tau s + 1}{(C_1 \to 0)s + [(C_2 - Ka) \to 0]} \to \infty \tag{6-48}$$

That is, the form of the series compensating network becomes indeterminant from the optimal control viewpoint, since $W_c = \infty$ requires infinite control deflections and the performance index no longer exists. Under this condition, the designer may try a different type of compensation, perhaps a feedback path compensating network or a combination of series and shunt compensation. It is easy to see that leaving the specification of the compensation to the ingenuity of the engineer could lead to serious difficulties, when the complexity of the problem becomes compounded by the presence of many output variables and many control variables.

Since the transfer functions which describe aircraft usually do not contain a free $1/s$ term, the second example will deal with the system

$$W(s) = \frac{K(as+1)}{bs^2 + cs + 1} \tag{6-49}$$

and the step input

$$R = \frac{1}{s} \tag{6-50}$$

The example yields results that lead to some interesting questions.

One now finds

$$\begin{aligned} N\bar{N} + k^2 D\bar{D} &= k^2b^2s^4 - (K^2a^2 + k^2c^2 - 2k^2b)s^2 + (K^2 + k^2) \\ &= (C_1s^2 + C_2s + C_3)(C_1s^2 - C_2s + C_3) \end{aligned} \tag{6-51}$$

After equating coefficients,

$$\begin{aligned} C_1 &= kb \\ C_2 &= \sqrt{K^2a^2 + k^2c^2 - 2k^2b + 2kb\sqrt{K^2 + k^2}} \\ C_3 &= \sqrt{K^2 + k^2} \end{aligned} \tag{6-52}$$

The expression for the optimal transfer function, Equation 6-33, is then

$$\begin{aligned} F_0 &= \frac{s[K(as+1)]}{(kbs^2 + C_2s + \sqrt{K^2+k^2})} \left[\frac{K(-as+1)}{s(kbs^2 - C_2s + \sqrt{K^2+k^2})} \right]_+ \\ &= \frac{K(as+1)}{kbs^2 + C_2s + \sqrt{K^2+k^2}} \cdot \frac{K}{\sqrt{K^2+k^2}} \\ &= \frac{\left(\frac{1}{1+\frac{k^2}{K^2}}\right)(as+1)}{\frac{kb}{\sqrt{K^2+k^2}}s^2 + \frac{C_2}{\sqrt{K^2+k^2}}s + 1} \end{aligned} \tag{6-53}$$

From Equation 6-53, one observes that the gain constant for the optimal closed-loop system is not equal to unity and only approaches unity as $k^2 \rightarrow 0$ (heavy weighting of the error term). The implication is that optimal control theory cannot design a closed-loop system with zero steady state error for a step input unless the free elements have an integrator. This is a serious difficulty from the viewpoint of the control engineer who designs stability augmentation systems for aircraft.

A second difficult point is encountered if one substitutes Equation 6-53 back into the performance index, Equation 6-11.

$$J_1 + k^2 J_2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}[F(s)-1][F(-s)-1]R(s)R(-s)ds$$
$$+\frac{k^2}{2\pi j}\int_{-j\infty}^{j\infty}\frac{F(s)F(-s)R(s)R(-s)}{W(s)W(-s)} \tag{6-54}$$

Both J_1 and $J_2 \rightarrow \infty$. That is, the performance index no longer exists and a question as to the validity of the design procedure is raised. In Equation 6-54, $J_1 \rightarrow \infty$ since F(s) - 1 has no free s to cancel R = 1/s. $J_2 \rightarrow \infty$ because F(s) / W(s) has no free s to cancel R = 1/s. That this is truly the situation is obvious since the error is finite and this in turn requires a finite control effort for all time. These two difficult points will be considered in the next sections.

6.5 THE PROBLEM OF A TYPE ZERO PLANT

As was seen in the preceding section, when the fixed elements are type zero and a step input is assumed, the performance index no longer exists. Hence, although the procedure yields a design, it is impossible to verify that it is an optimum one. However, it can be verified that the optimal design procedure yields a system which minimizes the steady state value of the integrand of the performance measure. This implies that it designs the system with the highest possible position constant (K_p) consistent with minimizing the integral square value of the control effort. Thus we conclude that the method is, at the very least, heuristically sound.

We now proceed with the proof that the $F_0(s)$ yielded by the design gives a steady state value for $(e^2 + k^2u^2)$ that is a minimum.

From Equation 6-32,

$$F_0(s) = \frac{W}{RY}\left[\frac{\tilde{W}R}{\tilde{Y}}\right]_+ \tag{6-55}$$

one finds

$$F_0(0) = \frac{1}{1+\dfrac{k^2}{W(0)^2}} \tag{6-56}$$

when R = 1/s.

Since $e(t) = \mathcal{L}^{-1}[(F-1)R]$ and $u(t) = \mathcal{L}^{-1}[FR/W]$, we may consider the expression

$$e^2 + k^2u^2 = \left\{\mathcal{L}^{-1}\left[\left[F(s)-1\right]\frac{1}{s}\right]\right\}^2 + k^2\left\{\mathcal{L}^{-1}\left[\frac{F(s)\,\frac{1}{s}}{W(s)}\right]\right\}^2$$

when $R(s) = 1/s$.

Expanding each term in a partial expansion gives:

$$\mathcal{L}^{-1}\left\{\left[F(s)-1\right]\frac{1}{s}\right\} = \left[F(s)-1\right] e^{st}\Big|_{s=0} \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \cdots$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{W(s)} \cdot \frac{1}{s}\right\} = \frac{F(s)}{W(s)} e^{st}\Big|_{s=0} \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \cdots$$

As $t \to \infty$, we obtain

$$\left[F(0)-1\right] = e(\infty)$$

and

$$F(0)/W(0) = u(\infty)$$

Therefore the steady state value of $(e^2 + k^2u^2)$ is

$$\left[F(0)-1\right]^2 + k^2 \left[\frac{F(0)}{W(0)}\right]^2 \tag{6-57}$$

Differentiating with respect to $F(0)$ and setting the resulting expression equal to zero then gives the value of $F(0)$ which gives the minimum steady state value for the integrand of the performance index. Performing this operation gives

$$F(0) = \frac{1}{1 + \frac{k^2}{W(0)^2}} \tag{6-58}$$

which agrees with Equation 6-56. Equation 6-56 was arrived at by using the equations for the design of the "optimum" system.

6.6 PROBLEM OF OBTAINING A "GOOD" LOW FREQUENCY RESPONSE WHEN NO FREE INTEGRATOR APPEARS

The problem of obtaining a closed-loop system with a zero steady state error response to a step input, when the plant has no free $1/s$ term, is now considered. The answer to the question of whether or not the optimal design procedure can design such a system is, surprisingly enough, a simple one. The optimal theory will design such a system if one properly defines the performance index.

This conclusion is a reasonable one, because one may argue that such a linear system can be designed without optimal control theory by simply "calibrating" the system beforehand. Now every linear system is an "optimum" one for some performance index, so we may argue that there is some

performance measure which will be a minimum for this system which has been calibrated to have zero steady state error.

6.7 THE PROBLEM OF THE FREE INTEGRATOR USING S. S. L. CHANG'S APPROACH

To prove the assertion, consider the problem of finding the control which minimizes the performance index of Equation 6-59 when one is given the system in Figure 18.

$$2V = \int_0^\infty \left[q(C_1 x_1 - C_2 x_2)^2 + ru^2\right] dt \tag{6-59}$$

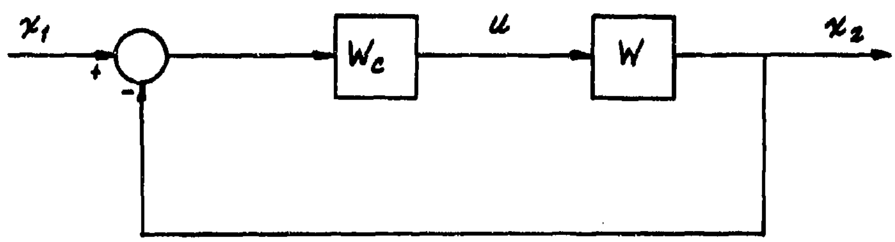


Figure 18. Single Output Optimal System

Applying Parseval's theorem to Equation 6-59 yields

$$2V = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[q(C_1 x_1 - C_2 x_2)(C_1 \bar{x}_1 - C_2 \bar{x}_2) + r u \bar{u}\right] dt \tag{6-60}$$

where

$$x_1 = x_1(s)\,,\ \bar{x}_1 = x_1(-s)\,,\ etc.$$

Observe that

$$x_2(s) = W(s)\,u(s) \tag{6-61}$$

It is assumed that the transfer function describing the fixed elements of the system has no free 1/s term. That is,

$$W(s) = \frac{K(a_0 s^n + a_1 s^{n-1} + \ldots + 1)}{b_0 s^m + b_1 s^{m-1} + \ldots + 1} \tag{6-62}$$

where m > n.

We may write Equation 6-60 in the following manner:

$$\frac{2V}{q} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[(C_1 x_1 - C_2 Wu)(C_1 \bar{x}_1 - C_2 \bar{W} \bar{u}) + \frac{r}{q} u \bar{u}\right] ds \tag{6-63}$$

Note: r/q is equivalent to the k^2 of the preceding section.

One now follows the usual procedure of substituting

$$u = u_o + \lambda u_1 \tag{6-64}$$

into Equation 6-63 and solving for the optimal control u_o. Carrying out this procedure, one can express Equation 6-63 in the form

$$\frac{2V}{q} = J_a + \lambda\left(J_b + J_c\right) + \lambda^2 J_d \tag{6-65}$$

where

$$J_a = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[\left(C_1 x_1 - C_2 W u_o\right)\left(C_1 \bar{x}_1 - C_2 \bar{W}\bar{u}_o\right) + \frac{r}{q}\, u_o \bar{u}_o\right] ds \tag{6-66}$$

$$J_b = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[\left(\frac{r}{q} + C_2^2 W\bar{W}\right)\bar{u}_o - C_1 C_2 W \bar{x}_1\right] u_1\, ds \tag{6-67}$$

$$J_c = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[\left(\frac{r}{q} + C_2^2 W\bar{W}\right)u_o - C_1 C_2 \bar{W} x_1\right] \bar{u}_1\, ds \tag{6-68}$$

$$J_d = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[\left(\frac{r}{q} + C_2^2 W\bar{W}\right)u_1 \bar{u}_1\right] ds \tag{6-69}$$

J_a is the optimum, $J_b = J_c$ and $J_d > 0$. Hence, the necessary and sufficient condition for $2V/q$ to be a minimum is

$$J_c = 0 \tag{6-70}$$

This is equivalent to saying

$$\left(\frac{r}{q} + C_2^2 W\bar{W}\right)u_o - C_1 C_2 \bar{W} x_1 = \eta(s) \tag{6-71}$$

must be analytic in the left-half plane. The solution to Equation 6-71 is (by direct comparison with Equations 6-28 and 6-31):

$$u_o = \frac{1}{\left[\frac{r}{q} + C_2^2 W\bar{W}\right]^+}\left[\frac{C_1 C_2 \bar{W} x_1}{\left[\frac{r}{q} + C_2^2 W\bar{W}\right]^-}\right]_+ \tag{6-72}$$

The optimal transfer function relating x_2 to x_1 is then

$$\frac{x_2}{x_1} = \frac{W}{x_1\left[\frac{r}{q} + C_2^2 W\bar{W}\right]^+}\left[\frac{C_1 C_2 \bar{W} x_1}{\left[\frac{r}{q} + C_2^2 W\bar{W}\right]^-}\right]_+ \tag{6-73}$$

For the special case where x_1 is a step, i.e.

$$x_1 = \frac{1}{s} \tag{6-74}$$

Equation 6-73 reduces to

$$\left.\frac{x_2}{x_1}\right|_{ss} = \frac{C_1 C_2 W(0)}{\left[\frac{r}{q} + C_2^2 W^2(0)\right]^+} \left[\frac{W(0)}{\left[\frac{r}{q} + C_2^2 W^2(0)\right]^-}\right]$$

$$= \frac{C_1 C_2 W^2(0)}{\frac{r}{q} + C_2^2 W^2(0)}$$

If we assume $C_2 = 1$, we then should pick

$$C_1 = 1 + \frac{r}{q} \cdot \frac{1}{W^2(0)} \tag{6-75}$$

in order that

$$\left.\frac{x_2}{x_1}\right|_{ss} = 1.0 \tag{6-76}$$

6.8 EQUIVALENCE BETWEEN THE TIME DOMAIN STATE VECTOR APPROACH AND THE FREQUENCY DOMAIN APPROACH

An equivalence exists between the state vector approach of Kalman and the frequency domain approach of Chang. The nature of this equivalence is best demonstrated by an example. For the example, we optimize (in the time domain) a second-order plant to a step input, to show that it is possible to obtain a control system whose steady state gain is unity, even though no free integrator appears in the plant. Thus the results of this section can be compared directly to those of the last section. In addition, the approach used to include the input in the set of first-order differential equations is a general one that will be given a more formal basis in the section on "Model Following" (Section 7.7).

It is possible to express a deterministic input in the form

$$\dot{x}_I = F_I x_I$$

where

x_I = state variables of the input
F_I = constant matrix.

The plant can be expressed in the form

$$\dot{x}_p = F_p x_p + G_p u_p$$

where

x_p = state vector of the plant
F_p = plant system matrix
G_p = plant input matrix
u_p = plant control vector

The input and the plant can be combined into a single representation and the combination can be optimized (see Equation 6-77).

$$\begin{bmatrix} \dot{x}_I \\ \dot{x}_p \end{bmatrix} = \begin{bmatrix} F_I & 0 \\ 0 & F_p \end{bmatrix} \begin{bmatrix} x_I \\ x_p \end{bmatrix} + \begin{bmatrix} 0 \\ G_p \end{bmatrix} \begin{bmatrix} u_p \end{bmatrix} \tag{6-77}$$

Now assume a step command input is applied to a closed-loop system which has the open-loop configuration shown below.

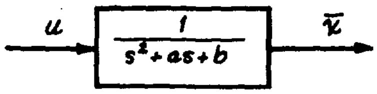


Figure 19. Block Diagram of $\frac{\bar{x}}{u}$

The step input can be described in the form

$$x_1 = 1, \ \dot{x}_1 = 0$$

The plant becomes:

$$x_2 = \bar{x}$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a x_3 - b x_2 + u$$

The entire system, consisting of input and plant, becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} u \end{bmatrix} \tag{6-78}$$

Suppose we decide to minimize on the difference between the weighted "output", $\bar{x} = x_2$, and the weighted step input, x_1. Choose

$$y = C_1 x_1 - C_2 x_2 \ ; \ H = \begin{bmatrix} C_1 & -C_2 & 0 \end{bmatrix} \tag{6-79}$$

where C_1 and C_2 are constants.

Let

$$\left.\begin{matrix} [R] = r \\ [Q] = q \end{matrix}\right\} \quad \text{r and q are scalars}$$

The performance index is

$$2V = \int_0^\infty \left[q(C_1 x_1 - C_2 x_2)^2 + r u^2 \right] dt \tag{6-80}$$

Substituting the H, Q, R, F and G matrices into the matrix Riccati equation (refer to Section 3.2):

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH \tag{6-81}$$

yields the expressions:

$$-\dot{p}_{11} = -\frac{p_{13}^2}{r} + q\,C_1^2 \tag{6-82}$$

$$-\dot{p}_{12} = -b p_{13} - \frac{p_{13}\,p_{23}}{r} - C_1 C_2\, q \tag{6-83}$$

$$-\dot{p}_{13} = p_{12} - a p_{13} - \frac{p_{13}\,p_{33}}{r} \tag{6-84}$$

$$-\dot{p}_{22} = -2b p_{23} - \frac{p_{23}^2}{r} + C_2^2\, q \tag{6-85}$$

$$-\dot{p}_{23} = p_{22} - a p_{23} - b p_{33} - \frac{p_{23}\,p_{33}}{r} \tag{6-86}$$

$$-\dot{p}_{33} = 2 p_{23} - 2a p_{33} - \frac{p_{33}^2}{r} \tag{6-87}$$

If one now solves for the values of the steady state gains using the expression

$$GK = GR^{-1}G'P \tag{6-88}$$

it is seen that the gains of interest are

$$\begin{aligned} K_1 &= \frac{p_{13}}{r} \\ K_2 &= \frac{p_{23}}{r} \\ K_3 &= \frac{p_{33}}{r} \end{aligned} \tag{6-89}$$

The over-all system is shown in Figure 20.

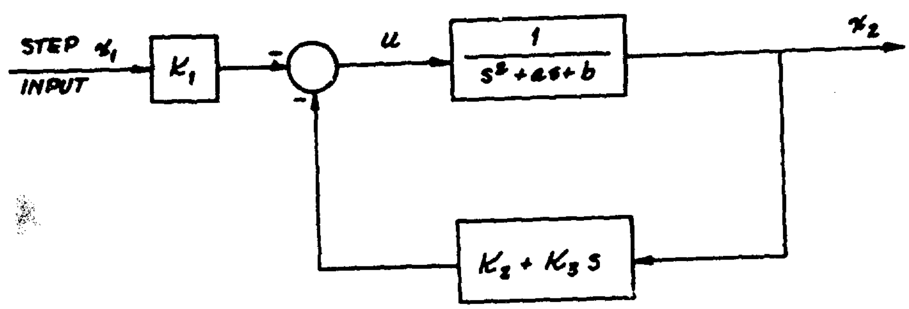


Figure 20. Feedback Configuration

The transfer function is

$$\frac{x_2}{x_1} = \frac{-K_1}{s^2 + (a + K_3) s + b + K_2}$$

$$= \frac{\frac{-K_1}{b+K_2}}{\left(\frac{s}{\sqrt{b+K_2}}\right)^2 + \left(\frac{a+K_3}{b+K_2}\right) s + 1} \qquad (6\text{-}90)$$

The remaining task is to select values of K_1, K_2, and K_3 that will yield

$$\left.\frac{x_2}{x_1}\right|_{ss} = 1.0 \qquad (6\text{-}91)$$

and are consistent with the set of Equations 6-82 - 6-87.

Let us first observe that the desire to have constant steady state gains K_1, K_2, and K_3 requires in turn that

$$\dot{p}_{13}, \dot{p}_{23}, \dot{p}_{33} \to 0 \qquad (6\text{-}92)$$

as $t \to \infty$. Thus it is necessary that Equations 6-84, 6-86 and 6-87 be set equal to zero. If Equation 6-84 is set equal to zero, then p_{12} must be a constant. Thus $\dot{p}_{12}$ is necessarily equal to zero and this forces Equation 6-83 to equal zero.

In a like manner, the fact that Equation 6-86 is equal to zero forces us to conclude that $\dot{p}_{22} \to 0$ as $t \to \infty$. Thus Equation 6-85 must also be set equal to zero.

We next observe that Equation 6-82 is an identity and is thus always satisfied, regardless of the value of p_{11} and can be safely excluded from the set of dependent equations.

If one solves Equation 6-85 for p_{23}, Equation 6-83 for p_{13} and Equation 6-87 for p_{33}, the resultant expressions will satisfy all the Equations 6-82 through 6-87.

Applying this procedure, one finds the following expressions for the steady state feedback gains:

$$K_1 = \frac{-C_1 C_2\, q/r}{\sqrt{b^2 + C_2^2\, q/r}} \tag{6-93}$$

$$K_2 = -b + \sqrt{b^2 + C_2^2\, q/r} \tag{6-94}$$

$$K_3 = -a + \sqrt{a^2 - 2b + 2\sqrt{b^2 + C_2^2\, q/r}} \tag{6-95}$$

The natural frequency of the system is defined by the expression

$$\omega_n^2 = \sqrt{b^2 + C_2^2\, q/r} \tag{6-96}$$

and the damping by the expression

$$\zeta = \frac{1}{2}\sqrt{\frac{a^2 - 2b + 2\sqrt{b^2 + C_2^2\, q/r}}{\sqrt{b^2 + C_2^2\, q/r}}} \tag{6-97}$$

From Equations 6-96 and 6-97, one observes that the natural frequency and damping are independent of the parameter C_1. On the other hand,

$$\left.\frac{x_2}{x_1}\right|_{ss} = \frac{-K_1}{b + K_2} \tag{6-98}$$

$$= \frac{C_1 C_2\, q/r}{b^2 + C_2^2\, q/r} \tag{6-99}$$

Thus the steady state value of x_2 (x_1 being a unit step) is directly proportional to C_1 and we can always force

$$\left.\frac{x_2}{x_1}\right|_{ss} = 1$$

For example, let $C_2 = 1$.

Then:

$$K_1 = \frac{-C_1\, q/r}{\sqrt{b^2 + q/r}}$$

$$K_2 = -b + \sqrt{b^2 + q/r}$$

$$K_3 = -a + \sqrt{a^2 - 2b + 2\sqrt{b^2 + q/r}} \tag{6-100}$$

and

$$\omega_n^2 = \sqrt{b^2 + q/r}$$

$$\zeta = \frac{1}{2}\sqrt{\frac{a^2 - 2b + 2\sqrt{b^2 + q/r}}{\sqrt{b^2 + q/r}}} \tag{6-101}$$

$$\left.\frac{x_1}{x_2}\right|_{ss} = \frac{C_1}{(r/q\, b^2 + 1)}$$

Therefore let

$$C_1 = 1 + \frac{r}{q} b^2 \tag{6-102}$$

and

$$\left.\frac{x_2}{x_1}\right|_{ss} = 1$$

That is, the steady state error $x_1 - x_2 = 0$.

Observe, from Equation 6-97, that as $q/r \rightarrow \infty$, the damping approaches .707 regardless of the value of C_2.

Notice that the choice of $C_2 = 1$ and $C_1 = 1 + (r/q)\, b^2$ gives us the same poles of the optimal system as one obtains using the performance index:

$$2V = \int_0^\infty \left[q(x_1 - x_2)^2 + r u^2\right] dt \tag{6-103}$$

Only the steady state gain constant is different.

In the previous section we obtained in Equation 6-75,

$$C_1 = 1 + \frac{r}{q}\,\frac{1}{W^2(0)}$$

Letting

$$W = \frac{1}{s^2 + as + b}$$

gives

$$W(0) = \frac{1}{b}$$

Therefore

$$C_1 = 1 + \frac{rb^2}{q}$$

This result, obtained using the frequency domain approach, agrees with Equation 6-102. The frequency domain answer was obtained in closed form (i.e., in terms of the transfer function) while the time domain answer comes out in terms of the various matrix entries.

As another example, suppose it is required that ω_n = constant. From Equation 6-96,

$$\omega_n^2 = \omega_o^2 = \sqrt{b^2 + C_2^2\, q/r}$$

pick

$$C_2^2 = \frac{\omega_o^4 - b^2}{q/r}$$

If we continue to require

$$\left.\frac{x_2}{x_1}\right|_{ss} = 1 = \frac{C_1 C_2\, q/r}{b^2 + q/r}$$

then pick

$$C_1 = \frac{\omega_o^4}{\sqrt{q/r\,(\omega_o^4 - b^2)}}$$

Using these values of C_1 and C_2 gives

$$K_1 = -\omega_o^2$$
$$K_2 = \omega_o^2 - b$$
$$K_3 = -a + \sqrt{a^2 + 2(\omega_o^2 - b)}$$

and a damping ratio of

$$\zeta = \sqrt{\frac{1}{2}\left(\frac{\omega_o^2 - b}{\omega_o^2}\right) + \frac{a^2}{4\omega_o}}$$

Thus $\zeta = .707$ when $\omega_o^2 >> b$. The system is given in Figure 21.

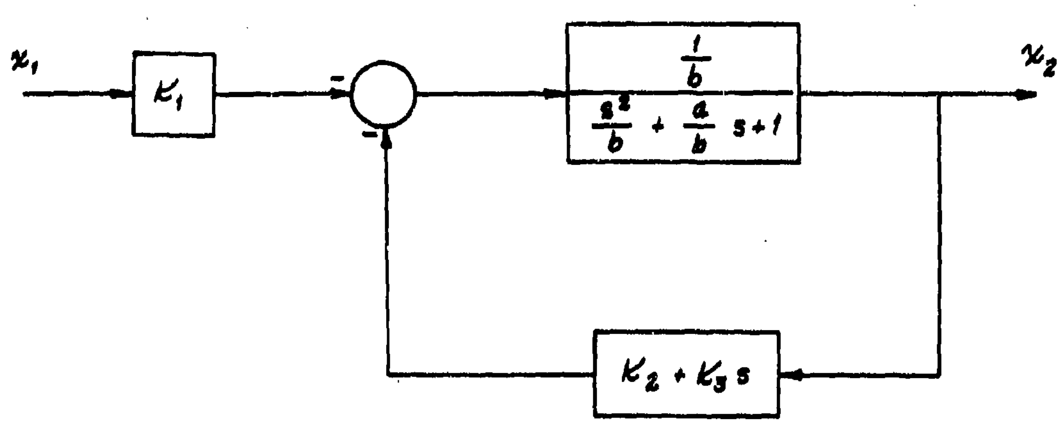


Figure 21. Closed-Loop Configuration

or

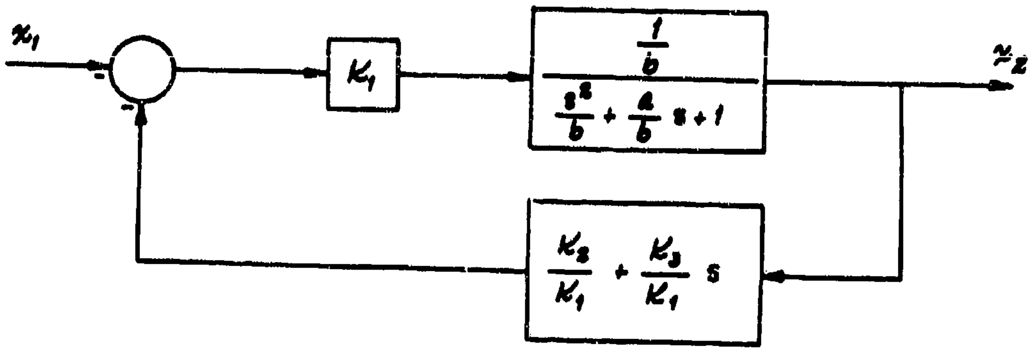


Figure 22. Alternate Feedback Configuration

where

$$\frac{K_2}{K_1} = \frac{b-\omega_0^2}{\omega_0^2}$$

$$\frac{K_3}{K_1} = \frac{a-\sqrt{a^2+2(\omega_0^2-b)}}{\omega_0^2}$$

The performance index is

$$2V = \int_0^\infty \left\{ q\left[\frac{\omega_0^4}{\sqrt{q/r\,(\omega_0^4-b^2)}}\, x_1 - \sqrt{\frac{\omega_0^2-b^2}{q/r}}\, x_2 \right]^2 + ru^2 \right\} dt$$

SECTION 7

MINIMIZATION IN THE FREQUENCY DOMAIN – THE MULTIVARIABLE PROBLEM

7.1 INTRODUCTION

In this section, the frequency domain approach of the previous section is extended to the multivariable case. The development is one which preserves the state space notation as much as possible but draws on the frequency domain approach of Chang for the basic analysis tools. It is shown that the frequency domain equation of interest, which was a scalar equation in Section 6, becomes a matrix equation of the Wiener-Hopf type. This equation is solved by two methods:

1. spectral factorization, or
2. a direct method.

Examples are given to demonstrate both approaches.

A subsection is included to show how one arrives at the matrix Wiener-Hopf equation when the basic description of the system is given in terms of transfer functions rather than a set of first-order differential equations.

The section concludes with a theoretical development of a method which uses a mathematical model to describe a desirable set of system dynamics and then requires the plant to match this set as closely as possible. Specifically, the model is included as a prefilter ahead of the plant and the object of the design is to move the plant poles to those regions of the s plane where they will not interfere with the model poles. Examples are given to illustrate the basic character of the results obtained and to demonstrate how one uses the optimal control law, $u_0 = -Kx$, to synthesize a specific closed-loop configuration.

7.2 THE REGULATOR PROBLEM

Consider the time domain vector equations

$$\left.\begin{aligned} y &= Hx \\ \dot{x} &= Fx + Gu + a(t) \end{aligned}\right\} \qquad (7\text{-}1)$$

where a(t) is a vector which takes into account any inputs to the system (e.g., a disturbance input).

The performance index is written as

$$2V = \int_0^{\infty} \left[y'(t) Q y(t) + u'(t) R u(t) \right] dt \qquad (7\text{-}2)$$

Applying Parseval's theorem to Equation 7-2 gives

$$2V = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left\{ y_* Q y + u_* R u \right\} dt \qquad (7\text{-}3)$$

where
$y = y(s)$

$\breve{y} = y(-s)$

$y_* = y'(-s)$ = transpose of the vector $y(-s)$

Taking the Laplace transform of Equation 7-1 yields:

$$y = H[Is-F]^{-1}Gu + H[Is-F]^{-1}[x(0)+a(s)] \tag{7-4}$$

For the sake of brevity, write

$$y(s) = W(s)u(s) + B(s)[x(0)+a(s)] \tag{7-5}$$

so that

$$\bar{y} = \bar{W}\bar{u} + \bar{B}\,[x(0)+\bar{a}] \tag{7-6}$$

and

$$y_* = (\bar{y})' = u_* W_* + [x'(0)+a_*]\,B_* \tag{7-7}$$

The expression for $y_* Qy$ becomes, after letting $B(s)[x(0)+a(s)] = B_1(s)$

$$y_* Qy = u_* W_* QWu + u_* W_* QB_1 + B_{1*} QWu + B_{1*} QB_1 \tag{7-8}$$

Substituting Equation 7-8 into Equation 7-3 gives

$$2V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{u_* W_* QWu + u_* W_* QB_1 + B_{1*} QWu + B_{1*} QB_1 + u_* Ru\right\}ds \tag{7-9}$$

We now seek the optimum control vector which minimizes the performance index of Equation 7-9. Let

$$u = u_0 + \lambda u_1 \tag{7-10}$$

where u_0 is the optimum component of u and u_1 is any arbitrary control vector which is analytic in the right-half plane. Substituting Equation 7-10 into Equation 7-9 gives:

$$\begin{aligned}2V = {} & \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left[u_{0*} W_* QWu_0 + u_{0*} W_* QB_1 + B_{1*} QWu_0 + B_{1*} QB_1 + u_{0*} Ru_0\right]ds \\ & + \frac{\lambda}{2\pi j}\int_{-j\infty}^{j\infty}\left[u_{0*} W_* QWu_1 + B_{1*} QWu_1 + u_{0*} Ru_1\right]ds \\ & + \frac{\lambda}{2\pi j}\int_{-j\infty}^{j\infty}\left[u_{1*} W_* QWu_0 + u_{1*} W_* QB_1 + u_{1*} Ru_0\right]ds \\ & + \frac{\lambda^2}{2\pi j}\int_{-j\infty}^{j\infty}\left[u_{1*} W_* QWu_1 + u_{1*} Ru_1\right]ds\end{aligned} \tag{7-11}$$

or

$$2V = J_a + \lambda(J_b + J_c) + \lambda^2 J_d \tag{7-12}$$

In Equation 7-12, J_a is the optimum component of 2V while J_d is always positive. On the other hand, the integrand of $J_b = J_c$ when $s \rightarrow -s$ and the necessary and sufficient condition for an optimum becomes (see, for example, page 14 of Reference 3):

$$J_c = 0 \tag{7-13}$$

The relationship of interest is

$$J_c = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} u_{1*} \left\{ \left[R + W_* Q W \right] u_o + W_* Q B_1 \right\} ds \tag{7-14}$$

Since u_{1*} is a vector which is analytic in the left-half plane, the condition

$$J_c = 0$$

can be expressed as

$$\left[R + W_* Q W \right] u_o + W_* Q B \left[x(0) + a(s) \right] = z(s) \tag{7-15}$$

where $z(s)$ must be analytic in the left-half plane.

For the sake of compactness, let

$$C = \left[R + W_* Q W \right] = R + G' \left[-Is - F' \right]^{-1} H' Q H \left[Is - F \right]^{-1} G \tag{7-16}$$

and

$$J = W_* Q B \left[x(0) + a(s) \right] = G' \left[-Is - F' \right]^{-1} H' Q H \left[Is - F \right]^{-1} \left[x(0) + a(s) \right] \tag{7-17}$$

Equation 7-15 becomes

$$C u_o + J = z \tag{7-18}$$

Equation 7-18 is a matrix equation of the Wiener-Hopf type which can be solved in two different ways. The method to be described first requires the factorization of the matrix C. The second approach takes advantage of the fact that the closed-loop poles of the optimal control law can be found from a scalar equation. Thus one need only solve Equation 7-18 for the zeros of u_o .

Thus far the solution of the matrix Wiener-Hopf equation for the optimal control vector has been emphasized. However, one may solve for the optimal output vector y_o under the special condition that the matrix of transfer functions is invertible (W^{-1} exists). A necessary condition for W^{-1} to exist is that the number of outputs equals the number of controls. When W^{-1} exists, the relationship

$$y = Wu + B_1$$

can be substituted into the matrix Wiener-Hopf equation to obtain either

$$R W^{-1} \left\{ \left[I + W R^{-1} W_* Q \right] y_o - B_1 \right\} = z$$

or

$$W_* \left\{ \left[Q + W_*^{-1} R W^{-1} \right] y_0 - W_*^{-1} R W^{-1} B_1 \right\} = z$$

as alternate expressions in terms of the optimal output y_0. In deriving the above results, it has been assumed that $(RW^{-1})^{-1} = WR^{-1}$.

$$[R + W_* Q W] u_0 + W_* Q B = z$$

$$y_0 = W u_0 + B_1$$

$$W^{-1} [y_0 - B_1] = u$$

$$[R + W_* Q W] [W^{-1} (y_0 - B_1)] + W_* Q B_1 = z$$

$$R [W^{-1} (y_0 - B_1)] + W_* Q (y_0 - \cancel{B_1}) + \cancel{W_* Q B_1} = z$$

$$[R W^{-1} + W_* Q] y_0 - R W^{-1} B_1 = z$$

$$R W^{-1} \left\{ \left[I + (R W^{-1})^{-1} W_* Q \right] y_0 - B_1 \right\} = z$$

if $(RW^{-1})^{-1} = WR^{-1}$, then

$$R W^{-1} \left\{ \left[I + W R^{-1} W_* Q \right] y_0 - B_1 \right\} = z$$

or

$$W_* \left\{ \left[Q + W_*^{-1} R W \right] y_0 - W_*^{-1} R W^{-1} B_1 \right\} = z$$

7.3 FACTORIZATION OF THE MATRIX

From Equation 7-16, one observes

$$C_* = [R + W_* Q W]_* = R + W_* Q W = C \qquad (7\text{-}19)$$

since R and Q are symmetrical.

Equation 7-19 is the defining relationship for a paraconjugate hermitian matrix and we are assured (Reference 8) that a factorization exists such that

$$C(s) = Y'(-s)\, Y(s) = Y_* Y \qquad (7\text{-}20)$$

where Y(s) is rational and analytic, together with its inverse Y^{-1}, in the right-half plane.

Substituting Equation 7-20 into Equation 7-18 gives:

$$Y_* Y u_o + J = \eta \qquad (7\text{-}21)$$

or:

$$Y_* \left[Y u_o + Y_*^{-1} J \right] = \eta \qquad (7\text{-}22)$$

where $Y_*^{-1} J$ may be decomposed into the sum

$$Y_*^{-1} J = \left[Y_*^{-1} J \right]_+ + \left[Y_*^{-1} J \right]_- \qquad (7\text{-}23)$$

in which the first factor on the right is analytic in the right-half plane and the second factor is analytic in the left-half plane.

Substituting Equation 7-23 into Equation 7-22 gives

$$Y_* \left[Y u_o + \left[Y_*^{-1} J \right]_+ + \left[Y_*^{-1} J \right]_- \right] = \eta \qquad (7\text{-}24)$$

Let

$$u_o = -Y^{-1} \left[Y_*^{-1} J \right]_+ \qquad (7\text{-}25)$$

and substitute above. We then have

$$Y_* \left[Y_*^{-1} J \right]_- = \eta$$

which is, by definition, analytic in the left-half plane. Thus we have satisfied the requirement that η be analytic in the left-half plane and verified that the assumed u_o is the optimal control vector.

The crucial step in solving for the optimal control vector is the task of spectral factoring the matrix C. In general, this is a tedious chore and one must resort to the algorithms given in References 8 and 9. The algorithm of Reference 9 permits one to draw some immediate conclusions concerning the form of the optimal control law and, when used in conjunction with other known facts about optimal systems, gives an indication of the amount of superfluous work required in the factorization approach.

From Reference 9 one finds the first step in factoring C to be the one of obtaining a least common denominator for all the entries of C. This step becomes trivial in view of the fact that all the transfer functions derived from a set of equations have the same poles. The least common denominator is the polynomial in s^2 which defines the open-loop roots and the conjugate of the open-loop roots. That is,

$$C = R + W_* Q W = \left[\frac{I}{D\bar{D}} \right] A \qquad (7\text{-}26)$$

where A is a p x p matrix which contains only polynomial entries (as opposed to the rational entries of R + W*QW) and D is the characteristic polynomial of the open-loop system.

Reference 9 further assures us that a factorization exists such that

$$\hat{A} - \hat{A}_{1*} \hat{A}_1 \tag{7-27}$$

where A_{1*} and A_1 have only polynomial entries. If we assume, for the sake of simplicity of presentation, that D is analytic in the right-half plane[†], it is seen that

$$Y = [I D^{-1}] A_1$$

and

$$Y_* = [I [\bar{D}]^{-1}] A_{1*}$$

Now

$$Y^{-1} = I D A_1^{-1} = \frac{I D A_1^{adj}}{|A_1|} \tag{7-28}$$

and

$$Y_*^{-1} = (I \bar{D}) A_{1*}^{-1} = \frac{I \bar{D} A_{1*}^{adj}}{|A_{1*}|} \tag{7-29}$$

Note that ID is a p x p matrix, where p is equal to the number of control variables and A_1^{adj} is again just a matrix with polynomial entries. Substituting the above expressions into the equation which defines the optimal control law (Equation 7-25) gives:

$$u_o = - \frac{I D [A_1^{adj}]}{|A_1|} \left[\frac{I \bar{D} A_{1*}^{adj}}{|A_{1*}|} \cdot W_* Q B_1 \right]_+ \tag{7-30}$$

Since A_{1*}^{adj} is a polynomial matrix, it has no poles which can contribute to the partial fraction expansion. The same is true of the $|A_{1*}|$, since it has only right-half plane roots. On the other hand, $W_* Q B_1$ has both left-half and right-half plane poles and will contribute to the partial fraction expansion (in particular, $W_* Q B_1$ has the poles of the open-loop system and the poles of a(s)). Thus the various components of Equation 7-30 must be of the form

$$\left.\begin{aligned} u_{o1} &= \frac{\xi_{o1}(s)}{(det.\, A_1)(LHP \text{ poles of } W_* Q B_1)} \\ u_{o2} &= \frac{\xi_{o2}(s)}{(det.\, A_1)(LHP \text{ poles of } W_* Q B_1)} \\ &\;\;\vdots \\ u_{on} &= \frac{\xi_{on}(s)}{(det.\, A_1)(LHP \text{ poles of } W_* Q B_1)} \end{aligned}\right\} \tag{7-31}$$

[†] When D has both right- and left-half plane poles, one uses the expressions

$$\{D\bar{D}\}^+ \quad \text{and} \quad \{D\bar{D}\}^-$$

where, for example, the $\{\ \}^+$ indicates the use of only the left-half plane roots.

In Equation 7-31, ξ_{o1}, ξ_{o2}, ... ξ_{on} are the polynomials which result when the partial fraction expansion entries of the left-half plane poles of $W_* Q B_1$, having been collected over a common denominator, are multiplied by $(-ID)A_1{}^{adj}$.

It is now desirable to obtain a relationship between the roots of $|A_1|$ and the roots of the multivariable root square locus, which are defined by the rational polynomial equation (refer to Equations 3-26 and 3-29).

$$|R + W_* Q W| = 0 \tag{7-32}$$

This relationship will give some indication of the superfluous effort involved in the factorization approach since the left-half plane roots of the root square locus have already been shown to be the only poles of the closed-loop system (refer to Section 3.2).

But

$$|R + W_* Q W| = \left|\left(\frac{I}{\bar{D}}\right) A_{1*} \left(\frac{I}{D}\right) A_1\right|$$

$$|R + W_* Q W| = \frac{\Delta \bar{\Delta}}{D \bar{D}} \tag{7-33}$$

where Δ defines left-half plane roots of the root square locus. Therefore

$$\frac{\Delta}{D}\left(\frac{\bar{\Delta}}{\bar{D}}\right) = \frac{|A_1|}{D^p} \cdot \frac{|A_{1*}|}{(\bar{D})^p} \tag{7-34}$$

$$|A_1| = \Delta D^{p-1}$$

u_{o1}, for instance, in Equation 7-31, would have the form

$$u_{o1} = \frac{\xi_{o1}(s)}{D^p \Delta \left(LHP \text{ poles of } a(s)\right)} \tag{7-35}$$

In Equation 7-35, we have extracted the D that would result from the partial fraction expansion and lumped it with D^{p-1}. Since it is already known that an open-loop root is not a closed-loop root it must be that $\xi_{o1}(s)$ contains the factor D^p. Thus, for example (in the regulator problem), three out of every four roots resulting from the factorization process must cancel when there are three controls (p = 3).

If we agree $\Gamma(s)$ = left-half plane poles of a(s) and that D^p has already been cancelled out of $\xi_{on}(s)$, then the components of u_o can be written in the compact form:

$$u_{on} = \frac{\xi_{on}(s)}{\Delta \Gamma} \tag{7-36}$$

where $\xi_{on}(s)$ is a polynomial of at least one order less than $\Delta\Gamma$. Thus all the poles of the optimal control are known both from inspection of the root square locus and the a(s) input vector. This fact can be used to good advantage to solve for u_o directly, without factoring C. The discussion of this application is postponed to Section 7.6.

One advantage of this procedure is apparent. It is seen that the factorization is independent of the input, since the a(s) vector is contained only in the J matrix. Thus there are no additional conceptual difficulties when there is an input to the system since the difficult portion of the over-all problem is the factorization of the rational matrix C.

In the next section it will be shown how one can formulate multivariable problems directly in the frequency domain after which an example will be given to illustrate the steps involved in solving Equation 7-25. The example is a contrived one which permits the factorization of the C matrix by inspection (i.e., the R matrix is singular).

7.4 DIRECT FORMULATION IN THE FREQUENCY DOMAIN IN TERMS OF TRANSFER FUNCTIONS

One often has the basic information concerning a multivariable system given in terms of transfer functions rather than F and G matrices. In this case, it is often advantageous to formulate the problem directly in the frequency domain.

Given the performance index

$$2V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\{y_* Q y + u_* R u\}\,ds \tag{7-37}$$

and

$$y(s) = W(s)\,u(s) + B(s) \tag{7-38}$$

one can proceed directly to the equation

$$[W_* Q W + R]u_o + W_* Q B = z \tag{7-39}$$

for which

$$u_o = -Y^{-1}\left[Y_*^{-1} J\right]_+ \tag{7-40}$$

and where

$$\left.\begin{aligned} C &= R + W_* Q W = Y_* Y \\ J &= W_* Q B \end{aligned}\right\} \tag{7-41}$$

Note that it is not necessary to think of y(s) as being defined in terms of F and G matrices. That is, given Equation 7-37 and a y(s), we may proceed directly to the matrix Wiener-Hopf equation (Equation 7-39). The factorization example of Section 7.5, the illustrative example of Section 7.6, and the model-following derivation of Section 7.7 should make this point clear.

7.5 FACTORIZATION EXAMPLE

Suppose we have the system depicted in Figure 23. There are two inputs to the system, $E_1(s)$ and $E_2(s)$. Our design objective is to minimize the difference between the actual system output and some desired output $E_3(s)$. In Figure 23, the transfer functions $w_1(s)$ and $w_2(s)$ represent

the fixed elements of the system, while W_{c_1} and W_{c_2} are the compensating networks which are to be designed. We limit the scope of the problem to solving for the two components of the optimal control vector, designated as u_{o_1} and u_{o_2}. This means that we are not particularly interested in what the system forcing functions ($E_1(s)$ and $E_2(s)$) are since they do not enter into the problem until the compensating networks are designed.

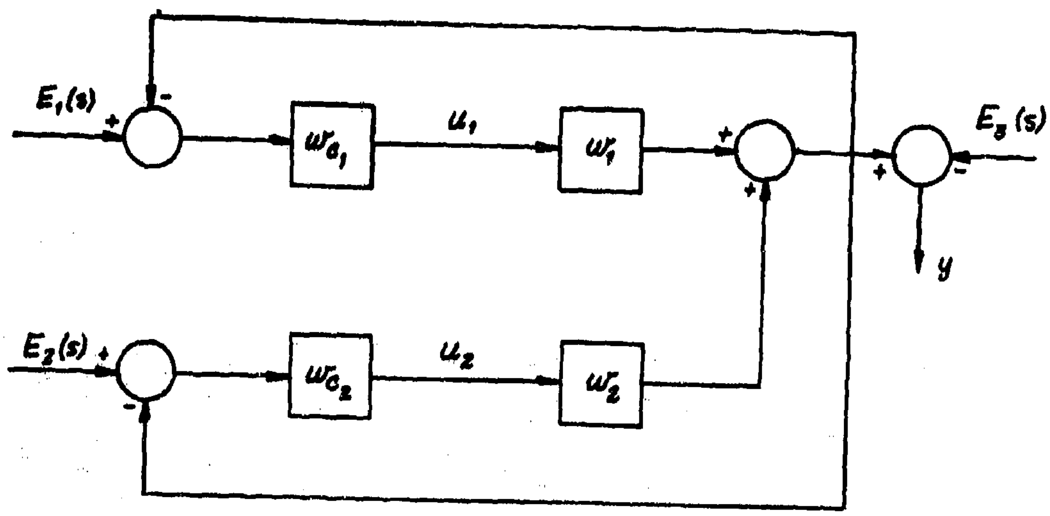


Figure 23. Block Diagram of System with Two Controls and a Single Output

The performance index is

$$2V = \int_0^\infty \{y^2 + (r_1 u_1 + r_2 u_2)^2\} dt = \int_0^\infty \left\{ y^2 + [u_1 \quad u_2] \begin{bmatrix} r_1^2 & r_1 r_2 \\ r_1 r_2 & r_2^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dt \tag{7-42}$$

From Figure 23, we see

$$y(s) = -E_3(s) + W_1 u_1 + W_2 u_2 = [W_1 \quad W_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - E_3 \tag{7-43}$$

which is a scalar.

Equations 7-42 and 7-43 tell us that

$Q = 1$ (a scalar)

$W = [W_1 \quad W_2]$, $W_* = \begin{bmatrix} \bar{W}_1 \\ \bar{W}_2 \end{bmatrix}$

$u_o = \begin{bmatrix} u_{o_1} \\ u_{o_2} \end{bmatrix}$

$B = -E_3$ (a scalar)

Substituting the above expression into Equation 7-39 gives

$$\left\{\begin{bmatrix}\bar{w}_1\\ \bar{w}_2\end{bmatrix} 1 \begin{bmatrix} w_1 & w_2\end{bmatrix} + \begin{bmatrix} r_1^2 & r_1 r_2 \\ r_1 r_2 & r_2^2\end{bmatrix}\right\}\begin{bmatrix}u_{o_1}\\ u_{o_2}\end{bmatrix} + \begin{bmatrix}\bar{w}_1\\ \bar{w}_2\end{bmatrix} 1 \begin{bmatrix}-E_3\end{bmatrix} = \begin{bmatrix} z_1 \\ z_2\end{bmatrix} \tag{7-44}$$

or

$$\begin{bmatrix} r_1^2 + w_1\bar{w}_1 & r_1 r_2 + \bar{w}_1 w_2 \\ r_1 r_2 + \bar{w}_2 w_1 & r_2^2 + w_2 \bar{w}_2\end{bmatrix}\begin{bmatrix}u_{o_1}\\ u_{o_2}\end{bmatrix} - \begin{bmatrix} E_3\bar{w}_1 \\ E_3 \bar{w}_2\end{bmatrix} = \begin{bmatrix} z_1 \\ z_2\end{bmatrix} \tag{7-45}$$

Therefore, C in Equation 7-41 is:

$$C = \begin{bmatrix} r_1^2 + \bar{w}_1 w_1 & r_1 r_2 + w_2\bar{w}_1 \\ r_1 r_2 + \bar{w}_2 w_1 & r_2^2 + w_2\bar{w}_2\end{bmatrix}, \quad D = \begin{bmatrix} -E_3\bar{w}_1 \\ -E_3\bar{w}_2\end{bmatrix}$$

or

$$C = \begin{bmatrix} r_1 & \bar{w}_1 \\ r_2 & \bar{w}_2\end{bmatrix}\begin{bmatrix} r_1 & r_2 \\ w_1 & w_2\end{bmatrix} = Y_* Y \tag{7-46}$$

In factoring C we have assumed w_1 and w_2 are transfer functions with left-half plane poles and zeros. Applying Equation 7-40, one finds

$$\begin{bmatrix}u_{o_1}\\ u_{o_2}\end{bmatrix} = \frac{-\begin{bmatrix} w_2 & -r_2 \\ -w_1 & r_1\end{bmatrix}}{(r_1 w_2 - r_2 w_1)}\left[\frac{\begin{bmatrix}\bar{w}_2 & -\bar{w}_1 \\ -r_2 & r_1\end{bmatrix}\begin{bmatrix}-E_3\bar{w}_1 \\ -E_3\bar{w}_2\end{bmatrix}}{(r_1\bar{w}_2 - r_2\bar{w}_1)}\right]_+$$

or

$$\begin{bmatrix}u_{o_1}\\ u_{o_2}\end{bmatrix} = \frac{-\begin{bmatrix} w_2 & -r_2 \\ -w_1 & r_1\end{bmatrix}}{(r_1 w_2 - r_2 w_1)}\left[\frac{\begin{bmatrix}-E_3\bar{w}_2\bar{w}_1 + E_3\bar{w}_1\bar{w}_2 \\ E_3 r_2\bar{w}_1 - E_3 r_1 \bar{w}_2\end{bmatrix}}{(r_1\bar{w}_2 - r_2\bar{w}_1)}\right]_+ \tag{7-47}$$

After simplifying Equation 7-47, assuming $E_3(s)$ has only left-half plane poles, one finds

$$\left.\begin{aligned} u_{o_1} &= \frac{-r_2 E_3(s)}{r_1 w_2(s) - r_2 w_1(s)} \\ u_{o_2} &= \frac{r_1 E_3(s)}{r_1 w_2(s) - r_2 w_1(s)}\end{aligned}\right\} \tag{7-48}$$

For this example, the root square locus expression becomes

$$\left| R + W_* Q W\right| = (r_1 w_2 - r_2 w_1)(r_1\bar{w}_2 - r_2\bar{w}_1) = 0 \tag{7-49}$$

One can also verify that

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} -\tilde{w}_1 [E_3]_- \\ -\tilde{w}_2 [E_3]_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7-50}$$

7.6 A DIRECT SOLUTION OF THE OPTIMAL CONTROL LAW

In this section, the matrix Wiener-Hopf equation (Equation 7-15 or 7-18) is solved directly by using the information contained in Equation 7-36:

$$u_{o_n} = \frac{\delta_n(s)}{\Delta\Gamma} \tag{7-51}$$

The advantage of the method is that Cramer's Rule can be used in solving for the optimal control law. The technique has obvious application in other fields where solutions to multidimensional Wiener-Hopf equations are required.

Briefly, the procedure is based on the fact that

$$(W_* Q W + R) u_0 + W_* Q B [x(0) + a(s)] = \gamma \tag{7-52}$$

is a vector which is analytic in the left-half plane. Since the poles of u_0 are already known from the root square locus, we may assume that the various components of the control vector can be written as polynomials (with unknown coefficients) divided by the closed-loop poles (and whatever poles that may enter due to forcing functions when the servo problem is being considered).

One then substitutes these expressions back into the vector equation

$$(W_* Q W + R) u_0 + W_* Q B_1 = \gamma \tag{7-53}$$

and forces a common denominator for the components of γ consisting of all the left-half plane poles. Since γ is analytic in the left-half plane, it must be that the numerator of Equation 7-53 has zeros that cancel the left-half plane poles. Thus when the least common left-half poles of Equation 7-53 are known, it is only necessary to let s take on values that force this denominator to zero. Since the numerator must also be zero, one obtains expressions that are equated to zero and yields equations in terms of the undetermined coefficients. There are usually more equations than unknowns but one will find that the correct number of these equations are linear combinations of the others and one is left with n linear equations in n unknowns. Two examples will demonstrate this and perhaps make the procedure clearer.

Suppose we have the configuration of Figure 23, and it is desired to minimize the performance index

$$2V = \int_0^\infty \left\{ y^2 + r_1 u_1^2 + r_2 u_2^2 \right\} dt = \int_0^\infty \left\{ y^2 + \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\} dt \tag{7-54}$$

This is the same example used in the previous section, but with a different performance index. The only difference lies in the R matrix, which

for this particular index, becomes:

$$\mathcal{R} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{7-55}$$

Corresponding to Equation 7-45 of the previous section, we have

$$\begin{bmatrix} r_1 + w_1 \bar{w}_1 & \bar{w}_1 w_2 \\ \bar{w}_2 w_1 & r_2 + \bar{w}_2 w_2 \end{bmatrix} \begin{bmatrix} u_{o_1} \\ u_{o_2} \end{bmatrix} - \begin{bmatrix} E_3 \bar{w}_1 \\ E_3 \bar{w}_2 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{7-56}$$

The root square locus is given by

$$r_1 r_2 + r_2 w_1 \bar{w}_1 + r_1 w_2 \bar{w}_2 = 0 \tag{7-57}$$

To demonstrate the technique described above, let

$$\left.\begin{aligned} E_3 &= \frac{1}{s} \\ w_1 &= \frac{s+2}{(s+1)(s+3)} \\ w_2 &= \frac{1}{(s+1)(s+3)} \\ r_1 &= \frac{1}{2} \\ r_2 &= \frac{1}{3} \end{aligned}\right\} \tag{7-58}$$

The expression for the root square locus becomes

$$\frac{(s+\sqrt{2})(s+\sqrt{10})(-s+\sqrt{2})(-s+\sqrt{10})}{6(s+1)(s+3)(-s+1)(-s+3)} = 0 \tag{7-59}$$

Therefore, the closed-loop roots are

$$s = -\sqrt{2}\ ,\ -\sqrt{10}$$

Assume

$$\left.\begin{aligned} u_{o_1} &= \frac{as^2+bs+c}{s(s+\sqrt{2})(s+\sqrt{10})} \\ u_{o_2} &= \frac{ds^2+es+f}{s(s+\sqrt{2})(s+\sqrt{10})} \end{aligned}\right\} \tag{7-60}$$

where a, b, c, d, e, and f are unknowns.

Substituting into Equation 7-57 gives

$$\frac{\left(\frac{s^4}{2}-6s^2+\frac{17}{2}\right)(as^2+bs+c)+(-s+2)(ds^2+es+f)-(-s+2)(s+3)(s+1)(s+\sqrt{2})(s+\sqrt{10})}{(s^2-4s+3)s(s+1)(s+3)(s+\sqrt{2})(s+\sqrt{10})}=\gamma_1 \tag{7-61}$$

$$\frac{-(s+2)(as^2+bs+c)+\left(\frac{s^4}{3}-\frac{10}{3}s^2+4\right)(ds^2+es+f)-(s+1)(s+3)(s+\sqrt{2})(s+\sqrt{10})}{(s^2-4s+3)s(s+1)(s+3)(s+\sqrt{2})(s+\sqrt{10})}=\gamma_2$$

Since γ_1 and γ_2 are analytic in the left-half plane, it must be that the numerator of each expression is zero when

$$s = 0,\ -1,\ -3,\ -\sqrt{2},\ -\sqrt{10}$$

This, of course, gives ten equations with only six unknowns. However, four equations turn out to be the same and we are left with the set:

$$\left.\begin{array}{ll}
\left.\begin{array}{l} 8.5c + 2f = 6\sqrt{20} \\ 2c + 4f = 3\sqrt{20} \end{array}\right\} & \text{when } s = 0 \\[2ex]
a - b + d - e = -\dfrac{21}{\sqrt{20}} & \text{when } s = -1 \\[2ex]
-3a + b + 3d - e = \dfrac{1}{\sqrt{20}} & \text{when } s = -3 \\[2ex]
-3a + \dfrac{3}{\sqrt{2}}b + (4+2\sqrt{2})d - (2+2\sqrt{2})e = \dfrac{-9}{\sqrt{10}} & \text{when } s = -\sqrt{2} \\[2ex]
-3a + \dfrac{3}{\sqrt{10}}b + (4+2\sqrt{10})d - \left(2+\dfrac{4}{\sqrt{10}}\right)e = \dfrac{-1.8}{\sqrt{2}} & \text{when } s = -\sqrt{10}
\end{array}\right\} \tag{7-62}$$

After using Cramer's Rule to solve for the unknowns, one finds

$$u_{01} = \frac{0.93147s^2 + 3.7323s + 12/\sqrt{20}}{s(s+\sqrt{2})(s+\sqrt{10})}$$

and

$$u_{02} = \frac{0.59037s^2 + 2.4853s + 9/\sqrt{20}}{s(s+\sqrt{2})(s+\sqrt{10})} \tag{7-63}$$

The output of the system,

$$y = \omega_1' u_{01} + \omega_2' u_{02} \tag{7-64}$$

is found, by direct substitution, to be

$$y = \frac{0.93147s + 2.45973}{s(s+\sqrt{2})(s+\sqrt{10})} \tag{7-65}$$

This concludes the first example. For the second example, a situation in which a multiple closed-loop root occurs will be worked out. This example will be carried out in greater detail since the manner in which the number of

equations reduces to the number of unknowns in the case of multiple roots is not as apparent as it is for simple roots (real or complex). For illustrative purposes, this example will be formulated from a "regulator" viewpoint (that is, the F and G matrices will be specified). Assume the matrices shown below are the given description of the system.

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathcal{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7-66}$$

Therefore,

$$\left[Is-F\right] = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}$$

and

$$\left[Is-F\right] = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix} \quad \text{and} \quad \left[Is-F\right]^{-1} = \frac{\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}}{s^2+4s+3}$$

$$\therefore \quad W = H\left[Is-F\right]^{-1}G = \frac{\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}}{s^2+4s+3}$$

and

$$W_* QW = \frac{\begin{bmatrix} -s+4 & -3 \\ 1 & -s \end{bmatrix}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}}{s^4-10s^2+9} = \frac{\begin{bmatrix} -s^2+25 & 4(-s+1) \\ 4(s+1) & -s^2+1 \end{bmatrix}}{s^4-10s^2+9} \tag{7-67}$$

Hence

$$\mathcal{R}+W_* QW = \frac{\begin{bmatrix} s^4-11s^2+34 & 4(-s+1) \\ 4(s+1) & s^4-11s^2+10 \end{bmatrix}}{s^4-10s^2+9} \tag{7-68}$$

The vector expression

$$\left[\mathcal{R}+W_* QW\right]u_0 + W_* QB\,x(0) = z(s) \tag{7-69}$$

becomes

$$\left.\begin{aligned} \frac{\left[(s^4-11s^2+34)u_{01}+4(-s+1)u_{02}+(-s^2+25)x_{01}+4(-s+1)x_{02}\right]}{s^4-10s^2+9} &= z_1 \\ \frac{\left[4(s+1)u_{01}+(s^4-11s^2+10)u_{02}+4(s+1)x_{01}+(-s^2+1)x_{02}\right]}{s^4-10s^2+9} &= z_2 \end{aligned}\right\} \tag{7-70}$$

The expression

$$|R + W_v Q W| = \frac{s^4 - 12s^2 + 36}{s^4 - 10s^2 + 9}$$

shows the closed-loop roots to be

$$\left\{s^4 - 12s^2 + 36\right\}^+ = \left(s + \sqrt{6}\right)^2 \tag{7-71}$$

Thus the expression

$$u_{0n} = \frac{\xi_n}{\Delta\Gamma}$$

yields

$$u_{01} = \frac{as + b}{(s + \sqrt{6})^2}, \quad u_{02} = \frac{cs + d}{(s + \sqrt{6})^2} \tag{7-72}$$

Substituting Equation 7-72 into Equation 7-70 and obtaining a least common denominator of the left-half plane poles gives the expressions

$$\left[\frac{(s^4 - 11s^2 + 34)(as + b) + 4(-s + 1)(cs + d) + \left[(-s^2 + 25)x_{01} + 4(-s + 1)x_{02}\right](s + \sqrt{6})^2}{(s^2 - 4s + 3)(s^2 + 4s + 3)(s + \sqrt{6})^2}\right] = z_1(s) \quad \text{(a)}$$

$$\text{(7-73)}$$

$$\left[\frac{4(s + 1)(as + b) + (s^4 - 11s^2 + 10)(cs + d) + \left[4(s + 1)x_{01} + (-s^2 + 1)x_{02}\right](s + \sqrt{6})^2}{(s^2 - 4s + 3)(s^2 + 4s + 3)(s + \sqrt{6})^2}\right] = z_2(s) \quad \text{(b)}$$

Now $z_1(s)$ and $z_2(s)$ must be analytic in the left-half plane. This is only possible if the numerators of Equation 7-73 contain

$$(s^2 + 4s + 3)(s + \sqrt{6})^2 \tag{7-74}$$

as a factor. Hence the numerator of both equations must equal zero when

$$s = -\sqrt{6}, -1, -3$$

Now consider, for example, the expression for $z_1(s)$ which is of the form

$$\frac{N_1(s)}{D_1(s)(s + \sqrt{6})^2} = z_1(s) \tag{7-75}$$

This implies

$$\frac{N_1(s)}{D_1(s)} = z_1(s)(s + \sqrt{6})^2 \tag{7-76}$$

Differentiating both sides of Equation 7-76 with respect to s gives

$$\frac{D_1 \frac{dN_1}{ds} - N_1 \frac{dD_1}{ds}}{D_1^2} = 2(s + \sqrt{6})z_1(s) + (s + \sqrt{6})^2 \frac{dz_1}{ds} \tag{7-77}$$

As $s \to -\sqrt{6}$ the right-hand side of Equation 7-77 $\to$ 0. Therefore,

$$D_1 \frac{dN_1}{ds} - N_1 \frac{dD_1}{ds} = 0$$

as $s \to -\sqrt{6}$. But $N_1(-\sqrt{6}) = 0$, therefore,

$$dN_1/ds = 0$$

as $s \to -\sqrt{6}$. One therefore concludes that the differential of each numerator of Equation 7-73 must equal zero for $s = -\sqrt{6}$.

We now proceed to sequentially let s take on the necessary values in Equation 7-73 (and in the differential of each numerator of Equation 7-73), and investigate the linear dependence of the expressions which result.

Let s = -1.

From Equation 7-73a, $-3a + 3b - c + d + \left[3x_{01} + x_{02}\right]\left[\sqrt{6} - 1\right]^2 = 0$ (7-78)

From Equation 7-73b, $0(-a+b) + 0(-c+d) + 0 = 0$

We now have one equation with four unknowns (Equation 7-78).

Let s = -3.

From Equation 7-73a, $16(-3a+b) + 16(-3c+d) + \left[16x_{01} + 16x_{02}\right]\left[\sqrt{6} - 3\right]^2 = 0$

From Equation 7-73b, $-8(-3a+b) - 8(-3c+d) + (-8x_{01} - 8x_{02})\left[\sqrt{6} - 3\right]^2 = 0$

These two equations are linearly dependent. Hence one obtains

$$-3a + b - 3c + d + \left[x_{01} + x_{02}\right]\left[\sqrt{6} - 3\right]^2 = 0 \qquad (7\text{-}79)$$

as the second equation in four unknowns.

Let s = $-\sqrt{6}$.

From Equation 7-73a, $4(-\sqrt{6}\,a + b) + 4(1 + \sqrt{6})(-\sqrt{6}\,c + d) = 0$

From Equation 7-73b, $4(1 - \sqrt{6})(-\sqrt{6}\,a + b) - 20(-\sqrt{6}\,c + d) = 0$

These two equations are linearly dependent. A third equation in four unknowns,

$$-\sqrt{6}\,a + b - (\sqrt{6} + 6)c + (1 + \sqrt{6})d = 0 \qquad (7\text{-}80)$$

has been obtained.

Differentiating the numerators of Equation 7-73 and letting s = $-\sqrt{6}$, gives two independent equations:

$$4a - \frac{\sqrt{6}}{2}b + \left[1 + 2\sqrt{6}\right]c - d = 0 \qquad (7\text{-}81)$$

and

$$\left(1-2\sqrt{6}\right)a+b-2c-\frac{\sqrt{6}}{2}d=0 \tag{7-82}$$

These equations are not linearly dependent and an apparent difficulty in having five equations with only four unknowns is resolved only when it is recognized that Equation 7-80 is a linear combination of Equations 7-81 and 7-82. Solving Equations 7-78, 7-79, 7-81 and 7-82 for a, b, c, and d (assuming $x_{01} = x_{02} = 1$) and substituting the results into Equation 7-72 gives:

$$u_{01} = \frac{-(0.8293s+3.6899)}{(s+\sqrt{6})^2}$$
$$u_{02} = \frac{-(0.20908s+.031326)}{(s+\sqrt{6})^2} \tag{7-83}$$

for the components of the optimal control vector. With this example, the section on finding the optimal control law in a direct fashion is concluded. It is seen that the basic feature which makes the method possible is the fact that if $\eta(s)$ is to be analytic in the left-half plane, it must contain zeros that cancel the left-half plane poles of $\eta(s)$. It is possible to write the components of $\eta(s)$ with a common left-half plane pole factor because the poles of u_0 are known (primarily from the root square locus).

7.7 MODEL FOLLOWING

The problem of including models in a system to approach a predefined set of closed-loop dynamics is an important concept in linear optimal control. Without a model, the poles of the closed-loop system will tend to adjust to a Butterworth pattern. Thus, while the Butterworth pattern is usually a desirable one, there may be cases in which a set of desired closed-loop poles are located in the s plane where they cannot be reached on the optimal root square locus. Of course, one could argue that if you already know where you want your closed-loop poles to be located, it is merely a matter of forcing the desired situation by solving for the necessary compensation. There is a serious flaw in an argument such as this because the optimal character of the solution, which places a constraint on the control, would be lost. A brute force solution for the desired compensation may well lead to control deflections which require impulses. On the other hand, the optimal control, which is always described by a proper rational function of s, does not admit impulses.

While one may usually think of the model-following concept as being used in such experiments where one aircraft is required to reproduce the dynamics of another (perhaps not yet built) aircraft, the concept can also be used to include specific inputs into the system. That is, one may consider the system inputs to be "models". (Another method for including inputs has already been discussed, namely, the a(t) vector of Equation 7-1.) It will be seen that the advantage of including inputs in the form of a prefilter model is that one may continue to use the feedback control law, $u_0 = -Kx$, to compensate the open-loop system.

To begin, it is assumed that the model is included in the physical system as a prefilter ahead of the plant.

The equations of the plant are taken as

$$\dot{x}_p = F_p x_p + G_p u \tag{7-84}$$

while the model dynamics are defined as

$$\dot{x}_m = L x_m \tag{7-85}$$

The state equation is now defined as a combination of the plant and model equations:

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_p \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & F_p \end{bmatrix} \begin{bmatrix} x_m \\ x_p \end{bmatrix} + \begin{bmatrix} 0 \\ G_p \end{bmatrix} [u] \tag{7-86}$$

This can be represented as

$$\dot{x} = Fx + Gu$$

When both the plant and model are of order n, and it is desired to compare all the state variables with all the model variables, a typical choice of the H matrix would be

$$H = \begin{bmatrix} I & -I \end{bmatrix}$$

an n x 2n matrix. When the plant and model are of unequal order, dummy variables can be added to the particular set that has the lowest order. That is, one of the matrices, either F_p or L, will pick up a zero row and a zero column for each dummy variable that is added. This defines the output vector which is to be minimized in the performance index as

$$y = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} x_m \\ x_p \end{bmatrix} = x_m - x_p \tag{7-87}$$

The Q matrix can be used to delete those error terms which one wishes excluded from the performance index. One may also specify zero terms in the H matrix that will immediately eliminate unwanted terms from the performance index.

The model has now been incorporated into the standard form of the regulator problem with the performance index defined as

$$2V = \int_0^T \left\{ y'Qy + u'Ru \right\} dt \tag{7-88}$$

with the optimal control law

$$u_0 = -R^{-1}G'P(t)x = -K(t)x$$

As $T \rightarrow \infty$, the feedback gain matrix becomes a matrix of constants, and the results of Section 7.4 can be used to solve for the optimal control law.

Taking the Laplace transform of Equation 7-87 and substituting in the

transforms of Equations 7-84 and 7-85, gives

$$y = -\left[Is - F_p\right]^{-1} G_p u - \left[Is - F_p\right]^{-1}\left(x_p(0) - \left[Is - F_p\right]\left[Is - L\right]^{-1} \quad {}_m(0)\right) \tag{7-89}$$

$$= Wu + B \tag{7-90}$$

as the equation which corresponds to Equation 7-38.

The Wiener-Hopf equation, Equation 7-39, becomes

$$\left(G_p'\left[-Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1} G_p + R\right) u_o$$
$$+ G_p'\left[-Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1}\left(x_p(0) - \left[Is - F_p\right]\left[Is - L\right]^{-1} x_m(0)\right) = z \tag{7-91}$$

Again one can write

$$Cu_o + J = z \tag{7-92}$$

and use the methods of the previous sections to solve for u_o.

Suppose now, for a given problem, that Equation 7-92 has been solved and the only remaining task is to force the open-loop control to obey the optimal control law. That is, we wish to find a specific feedback configuration for the system. The optimal control law

$$u_o = -K x(s) \tag{7-93}$$

can be used for this purpose if one is careful to remember that we are working with the standard form of the regulator problem. This means

$$x(s) = \begin{bmatrix} x_m(s) \\ x_p(s) \end{bmatrix} \tag{7-94}$$

which says that both the model variables and the plant variables are state variables. Since it is already known from Equation 7-93 that a feedback from each state variable to each control variable is required, one may think of the gains from the model state variables to the control variables as being feed-forward gains and the gains from the plant variables to the control variables as being feedback gains.

Notice that the direct method for solving a matrix Wiener-Hopf equation tells us that the poles of a model-following system are

1. the poles of the model
2. the poles found from the root square locus, which is found from the expression

$$\left| R + W_* Q W \right| = \left| R + G'\left[-Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1} \right| \tag{7-95}$$

This expression is independent of the model. The implementation of this portion of the model-following system therefore depends solely on the feedback gains.

The specific feedback configuration for the system can be found by using Equation 7-93 and implementing the following procedure:

1. The components of $x(s)$ are found, as specific functions of s, by substituting the optimal control solution into the expression

$$x(s) = \left[Is - F\right]^{-1} Gu + \left[Is - F\right]^{-1} x(0) \tag{7-96}$$

2. One now substitutes the specific components of $x(s)$, found from step 1 above, into the right-hand side of Equation 7-93.

3. The closed form expression for u_0 is then substituted into the left-hand side of Equation 7-93.

4. One can now solve for the unknown feedback gains since the two sides of the equation must be equal. The set of equations which results will always be a linear set.

Once again, it is emphasized that equations such as 7-96 and 7-93 involve <u>both</u> the model variables and the plant variables.

Two examples will serve to clarify the preceding discussion. In the first one, we wish to design an optimal second-order system which is to follow a step input. One may consider the step input to be represented by a model.

Let
$$x_{m1} = 1, \quad \dot{x}_{m1} = 0 \tag{7-97}$$

The matrix representation is

$$\dot{x}_{m1} = 0\, x_{m1} \tag{7-98}$$

The fixed elements (plant) are defined by Figure 24.

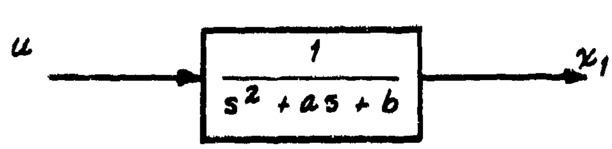


Figure 24. Open-Loop Plant

In Figure 24, x_1 is defined as the output variable. This system can be represented by the equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{7-99}$$

where

$$\dot{x}_1 = x_2$$

Therefore

$$F_p = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

The F_p matrix is 2 x 2 while the L matrix is 1 x 1 (see Equation 7-98). This is adjusted by adding the dummy model variable x_{m2} to Equation 7-98:

$$\begin{bmatrix} \dot{x}_{m1} \\ \dot{x}_{m2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{m1} \\ x_{m2} \end{bmatrix} \tag{7-100}$$

Thus

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$y = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} x_m \\ x_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{m1} \\ x_{m2} \\ x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} (x_{m1} - x_1) \\ (x_{m2} - x_2) \end{bmatrix} \tag{7-101}$$

We wish to exclude $x_{m2} - x_2$ from the performance index since our objective is to minimize the difference between x_1 and the step input. This is achieved by selecting

$$Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \tag{7-102}$$

R = r, a scalar, since there is only one control.

We may now proceed directly to Equation 7-91 and evaluate the various quantities of interest:

$$G_p'\left[-Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1} G_p + R = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} -s+a & -b \\ 1 & -s \end{bmatrix}}{s^2 - as + b} \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+a & 1 \\ -b & s \end{bmatrix}}{s^2 + as + b} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r$$

$$= r + \frac{q}{(s^2 - as + b)(s^2 + as + b)} \tag{7-103}$$

Assuming $x_p(0) = 0$, one finds (again referring to Equation 7-91):

$$-G_p' \left[-Is - F_p'\right]^{-1} Q \left[Is - L\right]^{-1} x_m(0) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} -s+a & -b \\ 1 & -s \end{bmatrix}}{s^2 - as + 1} \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}{s^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\frac{q}{s(s^2 - as + 1)}$$

Substituting into Equation 7-91 and dividing by q, one obtains:

$$\left[\frac{r}{q} + \frac{1}{(s^2 - as + b)(s^2 + as + b)}\right] u_0 - \frac{1}{s(s^2 - as + b)} = z_1$$

where

$$z_1 = \frac{z}{q}$$

or

$$\frac{r}{q}\left[\frac{(s^2 + \alpha s + B)(s^2 - \alpha s + B)}{(s^2 + as + b)(s^2 - as + b)}\right] u_0 - \frac{1}{s(s^2 - as + b)} = z_1 \qquad (7\text{-}104)$$

where

$$\alpha^2 = 2(B - b) + a^2$$

$$B = \sqrt{b^2 + q/r}$$

Using Equation 7-40,

$$u_0 = -Y^{-1}\left[Y_*^{-1} J\right]_+ \qquad (7\text{-}105)$$

and comparing 7-103 with 7-21, we find

$$Y^{-1} = \frac{s^2 + as + b}{\sqrt{r/q}\,(s^2 + \alpha s + B)}$$

$$Y_*^{-1} = \frac{s^2 - as + b}{\sqrt{r/q}\,(s^2 - \alpha s + B)}$$

and

$$J = \frac{-1}{s(s^2 - as + b)}$$

Equation 7-105 becomes

$$u_0 = \frac{\left(\frac{q}{r}\right)(s^2 + as + b)}{Bs(s^2 + \alpha s + B)} \qquad (7\text{-}106)$$

To find a specific feedback configuration, let

$$u_0 = -Kx = -K_1 x_{m1} - K_2 x_{m2} - K_3 x_1 - K_4 x_2$$

Applying Equation 7-96,

$$x(s) = [Is-F]^{-1}Gu + [Is-F]^{-1}x(0)$$

gives

$$\begin{bmatrix} x_{m1} \\ x_{m_2} \\ x_1 \\ x_2 \end{bmatrix} = \frac{\begin{bmatrix} 0 \\ 0 \\ s^2 \\ s^3 \end{bmatrix} u + \begin{bmatrix} s(s^2+as+b) \\ 0 \\ 0 \\ 0 \end{bmatrix}}{s^2(s^2+as+b)} \tag{7-107}$$

or

$$x_{m_1} = \frac{1}{s}$$

$$x_{m_2} = 0$$

$$x_1 = \frac{u}{s^2+as+b} = \frac{(q/r)}{B}\,\frac{1}{s(s^2+a^s+b)}$$

$$x_2 = \frac{su}{s^2+\alpha s+b} = \frac{(q/r)}{B}\,\frac{1}{s^2+\alpha s+B}$$

Therefore,

$$\frac{(q/r)}{B}\,\frac{(s^2+as+b)}{s(s^2+\alpha s+B)} = -\left[\frac{K_1}{s} + K_2\left(\cancel{x_{m2}}^{\,0}\right) + K_3\,\frac{(q/r)}{B}\,\frac{1}{s(s^2+\alpha s+B)} + K_4\,\frac{(q/r)}{B}\,\frac{1}{(s^2+\alpha s+B}\right] \tag{7-108}$$

After placing the right-hand side of Equation 7-108 over a least common denominator, one can equate coefficients in the resultant polynomials and solve for the K's. The result is

$$\begin{aligned} K_1 &= -\frac{q/r}{\sqrt{b^2+q/r}} \\ K_2 &= 0 \\ K_3 &= -b+\sqrt{b^2+q/r} \\ K_4 &= -a+\sqrt{a^2+2\left(\sqrt{b^2+q/r}-b\right)} \end{aligned} \tag{7-109}$$

A specific system configuration is shown in Figure 25.

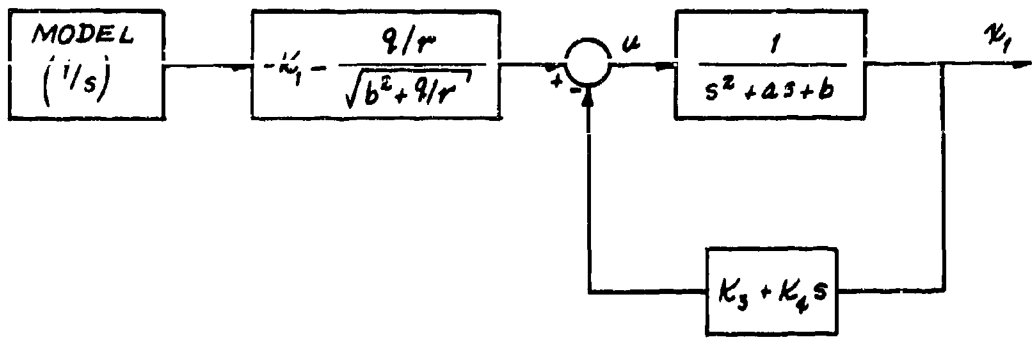
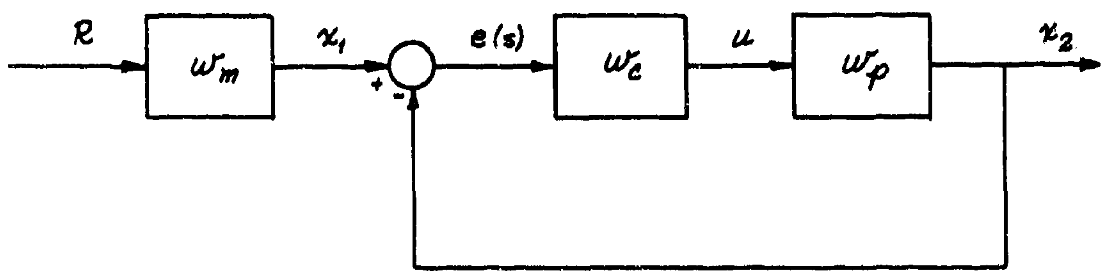


Figure 25. A Specific Feedback Configuration

A more elaborate example of the model-following technique is given in Section 9.

We close with a simple example designed to show the basic consequence of the generally accepted model-following performance index and to further demonstrate the applicability of the method outlined in Section 6.3. Consider the single output situation depicted in Figure 26.



W_m = transfer function of the model

W_p = transfer function of the plant

W_c = compensating network

Figure 26. Single-Input, Single-Output Model-Following System

The performance index is

$$2V = \int_0^{\infty} \left[q(x_1 - x_2)^2 + ru^2 \right] dt \qquad (7\text{-}110)$$

Applying the results of Section 6.3, one finds

$$u_0 = \frac{1}{\left[\frac{r}{q} + w_p \bar{w}_p\right]^+} \left[\frac{R \bar{w}_p w_m}{\left[\frac{r}{q} + w_p \bar{w}_p\right]^-} \right]_+ \qquad (7\text{-}111)$$

as the optimal control vector. The optimal transfer function of $x_2/R\ (s)$ is

$$\frac{x_2}{R}(s) = \frac{w_p}{R\left[\frac{r}{q} + w_p \bar{w}_p\right]^+} \left[\frac{R \bar{w}_p w_m}{\left[\frac{r}{q} + w_p \bar{w}_p\right]^-} \right]_+ \qquad (7\text{-}112)$$

As $q \to \infty$, we see that the optimal transfer function tends toward

$$\frac{x_2}{R} = \frac{1}{R} \left[R w_m \right]_+ \qquad (7\text{-}113)$$

Suppose R = 1, (R(t) is a delta function) and w_m has all left half plane poles, then

$$x_2 = w_m = x_1 \qquad (7\text{-}114)$$

That is, when $r/q = 0$, the output follows the input exactly. This result in no way depends on the form of w_m and w_p.

When R is a step,

$$\frac{x_2}{R} = \left[w_m(0) \right] + s \left[\frac{1}{s} w_m \right] \text{ evaluated at the poles of } w_m$$

$$= \frac{\xi_1(s)}{\text{poles } w_m} \qquad (7\text{-}115)$$

That is, with a step input and $r/q = 0$, the optimal transfer function is not exactly equal to w_m but does have the same poles as w_m.

In the more general situation, when $r/q \neq 0$, the expression for the optimal transfer function indicates a form

$$\frac{x_2}{R} = \frac{\xi_2(s)}{\text{(poles of root square locus) (poles of model)}} \qquad (7\text{-}116)$$

Thus the poles of the model are always present, and clearly one of the design problems is to pick an r/q that will yield root square locus closed-loop poles far enough displaced from the poles of the model so that the dominant roots are those of the model. This viewpoint may result in a set of feedback gains which are high enough to cause practical difficulties when the plant is an aircraft. This point will be discussed in the more elaborate model-following example of Section 9.4. It is also apparent that the effect of r/q on $\xi_z(s)$ cannot be determined from the root square locus alone.

The previous example was selected to highlight what appears to be the basic characteristic of the pole pattern of the optimal system when the model-following scheme is employed - that is, the poles of the model are always present while the poles of the plant approach a Butterworth configuration as they migrate more deeply into the left half plane (or encounter a zero of the open-loop system). In addition, the zeros of the system are not defined by the root square locus alone.

SECTION 8

USE OF BODE PLOTS IN LINEAR OPTIMAL DESIGN

8.1 INTRODUCTION

Bode plots are particularly well suited for the task of determining the closed-loop poles of the optimal system as a function of the weighting parameters of the performance index. The use of Bode plots is first outlined for the single-input, single-output problem and then extended to the multivariable case. A relatively complicated design problem, involving a jet fighter in a power approach, is used to illustrate the application of the concept.

All the basic tools for a frequency domain design procedure are available after the use of the Bode plots is outlined. The section concludes with an outline of a design procedure which utilizes these tools and can be carried out entirely in the frequency domain.

8.2 SINGLE VARIABLE CASE

The purpose of the multivariable root square locus is to determine the closed-loop poles of the optimal system as a function of the weighting factors of the performance index. Bode plots are also particularly well suited to this task. The basic reason for this is that in the root square locus only the products of a transfer function with its conjugate are involved [e.g., $W(s)W(-s)$]. Hence, when $s \rightarrow j\omega$ the phase is always identically equal to zero. This means that the closed-loop roots can always be found from the open-loop transfer function using only the zero degree line on a Nichols Chart.* Hence accurate frequency domain representations of the closed-loop optimal poles are easily found.

While accurate mathematical descriptions of the closed-loop poles are possible through the use of the Nichols Chart, in practice one finds that the usual open-to-closed loop approximations used in servo work become quite accurate when applied to optimal control. Since one is in essence working with a $|W|^2$ type equation, the slopes encountered at breakpoints are always on the order of 40 db/decade at a minimum. Since the maximum possible error encountered in using the straight line approximation is 6 db, and since we are working with steep slopes, the error in the break frequencies due to the use of the straight line approximation is small.

The essence of the Bode plot method lies in recognizing that the expression

$$|R + W_{*} Q W| \tag{8-1}$$

can always be placed in a multi-unity feedback block diagram form. It is only necessary to demonstrate this for a few typical cases after which the technique will become quite transparent. To demonstrate, we consider first the simplest of cases. For the single-input, single-output problem, the equation

---

* When a particular value of q or r is permitted to take on a negative value, the 180° line on the Nichols Chart must be used.

$$|R + W_u Q W| = 0 \tag{8-2}$$

reduces to

$$1 + \frac{q}{r} w\bar{w} = 0 \tag{8-3}$$

Equation 8-3 can be investigated in two ways by placing it in unity feedback form. First, consider the block diagram of Figure 27.

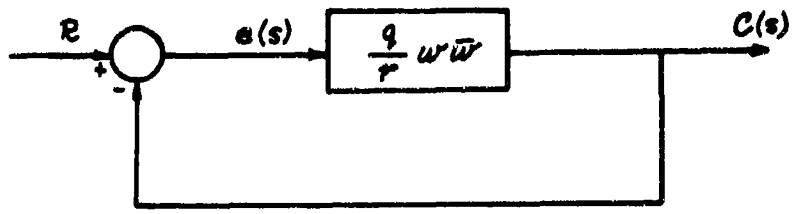


Figure 27. First Unity Feedback Form

Using block diagram algebra, one easily finds

$$\frac{C}{R} = \frac{\frac{q}{r} w\bar{w}}{1 + \frac{q}{r} w\bar{w}} \tag{8-4}$$

Thus the closed-loop poles of the system shown in Figure 27 are the same as the roots of Equation 8-3.

The second way of treating Equation 8-3, and experience shows it to be the preferable way, is to use the unity feedback form shown in Figure 28.

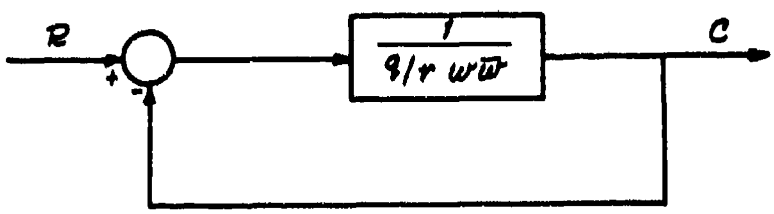


Figure 28. Second Unity Feedback Form

The transfer function is now

$$\frac{C}{R} = \frac{1}{1 + \frac{q}{r} w\bar{w}} \tag{8-5}$$

The closed-loop poles of the system shown in Figure 28 are the same as the roots of Equation 8-3. The advantage of this second approach will become apparent in the multi-input situation.

When using Bode plots,

$$\left.\frac{1}{w\bar{w}}\right|_{s=j\omega} = \left.\frac{1}{w(s)\,w(-s)}\right|_{s=j\omega} = \left|\frac{1}{w(j\omega)}\right|^{2} \qquad (8\text{-}6)$$

One has the option of either plotting the open-loop transfer function with double breaks at the break frequencies or plotting the open-loop function and changing the scale of the ordinate.

For those frequencies for which

$$\frac{1}{q/r\; w\bar{w}} \gg 1, \qquad (8\text{-}7)$$

$$\frac{C}{R} \doteq 1.0 \qquad (8\text{-}8)$$

For those frequencies for which

$$\frac{1}{q/r\; w\bar{w}} \ll 1, \qquad (8\text{-}9)$$

$$\frac{C}{R} \doteq \frac{1}{q/r\; w\bar{w}} \qquad (8\text{-}10)$$

To find the optimal closed-loop poles, one simply plots the open-loop response with $q/r = 1$. The poles associated with other values of q/r are found by sliding the open-loop plot up or down by the required number of decibels. The breaks read from the Bode plot represent both the left-half and image right-half plane break frequencies. The left-half plane component is, of course, the only one of interest. The approximations given by Equations 8-8 and 8-10 have proven to be quite accurate for the majority of purposes. Of course, if one prefers, the "0° line" on a Nichols Chart can be used to obtain more exact results. For convenience, an expanded plot of this "0° line" is given in Figure 29.

There may be, depending on the form of w, a family of values of r/q for which either Equation 8-8 holds or Equation 8-10 holds. This implies that a range of r/q exists for which the closed-loop roots will not be altered significantly. When this situation exists, the Bode plot approach gives the range of r/q immediately. The use of this property will be more clearly demonstrated by the power approach example of Section 8.5 and the more elaborate model-following example of Section 9.3.

8.3 EXTENSION TO MULTIVARIABLE CASE

This procedure is readily extended to the multivariable problem. The equation

$$|R + W_* Q W| = 0$$

in a typical two control variable and two output variable problem might reduce to

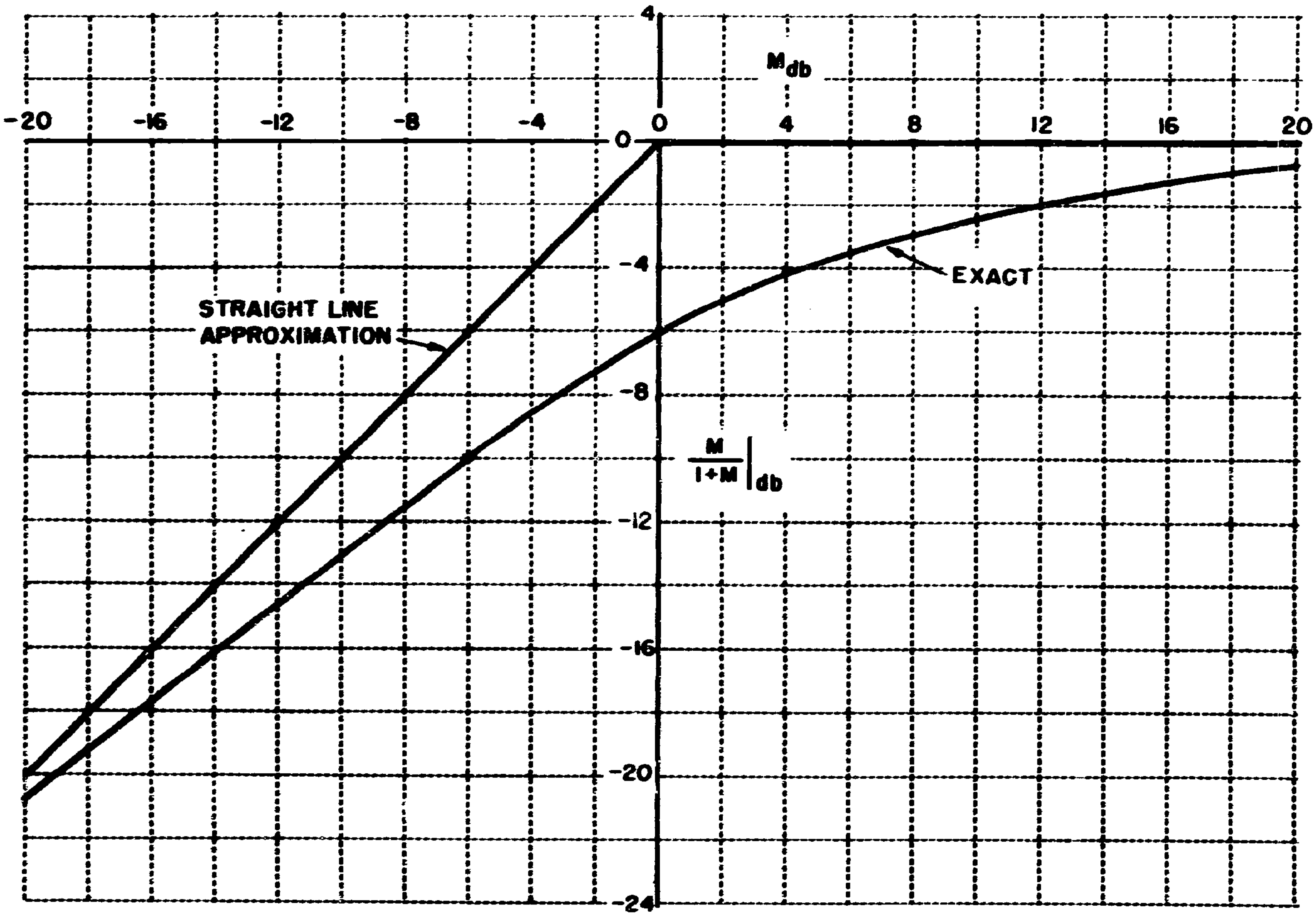


Figure 29. Expanded Plot of the "0° Line" on a Nichols Chart

$$1+\frac{q_1}{r_1}\,w_{11}\bar{w}_{11}+\frac{q_1}{r_2}\,w_{12}\bar{w}_{12}+\frac{q_2}{r_1}\,w_{21}w_{21}+\frac{q_2}{r_2}\,w_{22}\bar{w}_{22}$$

$$+\frac{q_1}{r_1}\,\frac{q_2}{r_2}\left[(w_{11}w_{22}-w_{12}w_{21})(\bar{w}_{11}\bar{w}_{22}-\bar{w}_{12}\bar{w}_{21})\right]=0 \qquad (8\text{-}11)$$

where

$$w_{11}=\frac{y_1}{u_1}(s),\ \bar{w}_{11}=\frac{y_1}{u_1}(-s),\ w_{12}=\frac{y_1}{u_2}(s),\ w_{21}=\frac{y_2}{u_1}(s),\ w_{22}=\frac{y_2}{u_2}(s)$$

To employ the Bode plot concept, set

$$(w_{11}w_{22}-w_{12}w_{21})=\begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix}=Z \qquad (8\text{-}12)$$

As shown in Appendix II, the proper power of the open-loop roots will always factor out of the numerator of expressions like Equation 8-12. Hence the order of the denominator of the rational expression Z is the same as the rest of the terms in Equation 8-11. Equation 8-11 becomes

$$1+\frac{q_1}{r_1}\,w_{11}\bar{w}_{11}+\frac{q_1}{r_2}\,w_{12}\bar{w}_{12}+\frac{q_2}{r_1}\,w_{21}\bar{w}_{21}+\frac{q_2}{r_2}\,w_{22}\bar{w}_{22}+\frac{q_1}{r_1}\,\frac{q_2}{r_2}\,Z\bar{Z}=0 \qquad (8\text{-}13)$$

If one uses the first approach outlined above to implement the unity feedback form, the result might be the block diagram of Figure 30. (NOTE: Other implementations are obviously possible by properly interchanging the blocks.)

When the block diagram of Figure 30 is analyzed (one must close the loop five times) the use of the open-to-closed loop approximations becomes invaluable due to the speed with which they can be implemented. A "serious" disadvantage of this particular block diagram implementation is that the open-loop poles enter immediately into the first block while all succeeding blocks deal only with ratios. Thus, for example, if the open-loop poles involve actuator dynamics, one is forced to include them in the first stage of the process and their effect is likely to be greatly obscured by the time the fifth stage is reached.

The more desirable implementation is given in Figure 31. In the implementation of Figure 31, only the zeros of the various transfer functions need be considered in the first four states. The open-loop poles of the system do not enter until the last stage.

A design problem which illustrates the use of Bode plots will be given in Section 8.5 after the steps in a design procedure which can be carried out entirely in the frequency domain are given in Section 8.4.

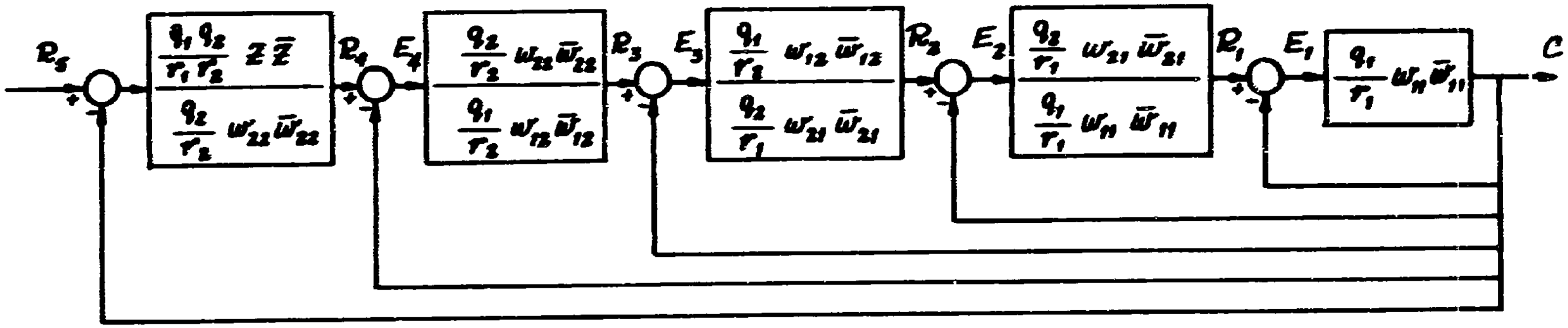


Figure 30. **First Feedback Form for Multivariable System**

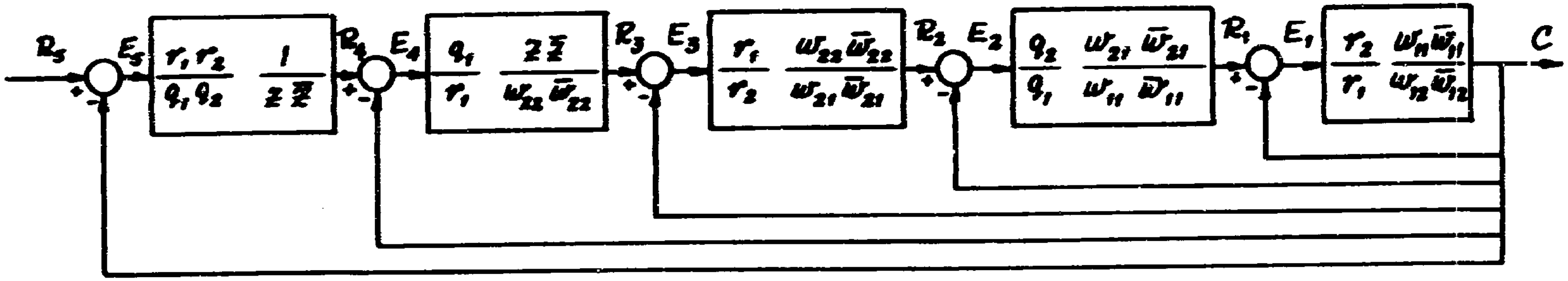


Figure 31. **Second Feedback Form for Multivariable System**

8.4 STEPS IN A FREQUENCY DOMAIN DESIGN PROCEDURE

It is now possible, using the results of the previous sections, to postulate a trial-and-error design procedure which can be carried out entirely in the frequency domain. The four steps are:

1. Using the multivariable root square locus (or Bode plots), find the closed-loop poles of the optimal system as a function of the weighting factors of the performance index. This is a trial and error procedure for there are many values of the q's and r's which may give the same closed-loop poles.

2. After deciding on a particular set of q's and r's from (1) above, solve for the components of the optimal control law. There are two ways to do this:
   a. spectral factorization (Section 7.3), or
   b. the direct method (Section 7.6).

3. Substitute u_0 back into the expression
   $$x(s) = [Is-F]^{-1}Gu + [Is-F]^{-1}x(0)$$
   One can now use the equation
   $$u_0 = -Kx(s)$$
   and a partial fraction expansion (or more direct algebraic means if desired) to solve for the unknown feedback gains (refer to the example in Section 7.7).

4. One may then repeat the first three steps using a different set of q's and r's (which give the same closed-loop poles) if the initial trial attempt yields a system for which the feedback gains are too high. An adjustment in the design objective is called for if one exhausts the possible values of q and r before obtaining a desirable set of feedback gains.

Of the four steps, it is the first which requires the greatest exercise of engineering judgment. This is primarily due to the large range over which the weighting factors may vary without any essential change in the closed-loop poles. The design undertaken in Section 9.3 affords an excellent illustration of the large range through which some of the weighting factors of the performance index may vary without any appreciable effect on the closed-loop poles. Thus good engineering judgment is required of the designer if he is to accurately specify the possible range of the parameter variation without the expenditure of an excessive amount of time.

8.5 DESIGN PROBLEM WITH BODE PLOTS

To illustrate the use of Bode plots in a more complex design situation, we consider the application of the model-following method of Section 7.7 to the problem of a high performance jet fighter in a power approach. The block diagram of Figure 32 will be used for this investigation. The numerical values for the various time constants, etc., are taken from Section 5.3.

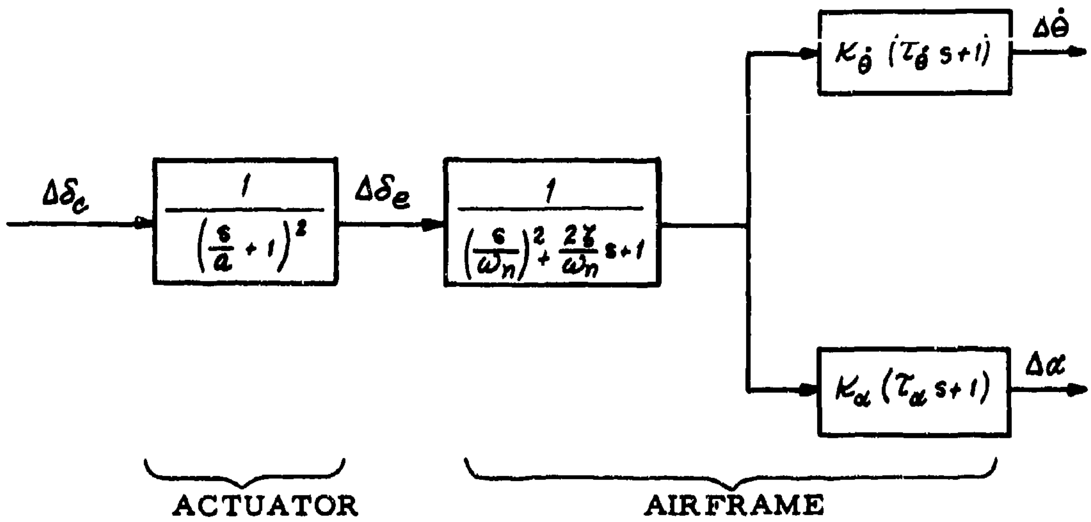


Figure 32. Block Diagram of Open-Loop System

In Figure 32, $\Delta\dot\theta$ and $\Delta\alpha$ are the variables to be controlled while $\Delta\delta_c$ is the stick deflection which (through the actuator) causes an elevator deflection, $\Delta\delta_e$. The only control variable is $\Delta\delta_c$. We seek the control law which will define the optimal $\Delta\delta_c$. The various feedback gains, which will force $\Delta\delta_c$ to behave optimally, have not been shown in Figure 32. Since our objective is to point out how one employs the Bode plots as a design aid, we will limit the scope of the problem to specifying approximate values of the parameters which will more than likely give good model-following in both $\Delta\alpha$ and $\Delta\dot\theta$. That is, the variation of the plant poles as a function of q_1, q_2, r and a (the actuator root) will be studied but the zeros of the optimal control law will not be computed nor will the feedback gains be specified.

As a performance index, we choose

$$2V = \int_0^\infty \left[q_1(\Delta\dot\theta_m - \Delta\dot\theta)^2 + q_2(\Delta\alpha_m - \Delta\alpha)^2 + r\Delta\delta_c^2\right]dt \tag{8-14}$$

Using the method of Section 7.4 (or the analysis technique of Section 6.3) one finds, letting $\delta_c = \delta_{c_0} + \delta_{c_1}$,

$$\left(1 + \frac{q_1}{r}w_{\dot\theta}\bar{w}_{\dot\theta} + \frac{q_2}{r}w_\alpha\bar{w}_\alpha\right)\delta_{c_0} - \left(\frac{q_1}{r}w_a\bar{w}_{\dot\theta} + \frac{q_2}{r}w_b\bar{w}_\alpha\right) = \mathcal{Z}(s) \tag{8-15}$$

must have right-half plane poles.

In Equation 8-15

$$\Delta\dot\theta_m(s) = w_a = \frac{(s+1.5)\Delta\dot\theta_m(0) - 14.5\Delta\alpha_m(0)}{s^2 + 5.4s + 20.35}$$

$$\Delta\alpha_m(s) = w_v = \frac{(s+3.9)\Delta\alpha_m(0)+\Delta\dot{\theta}_m(0)}{s^2+5.4s+20.35}$$

$$\frac{\Delta\dot{\theta}}{\Delta\delta_c} = w_{\dot{\theta}} = \frac{K_{\dot{\theta}}(\tau_{\dot{\theta}}s+1)}{\left[\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1\right]}\;\frac{1}{\left(\frac{s}{a}+1\right)^2} = \frac{-1.175\left(\frac{s}{.495}+1\right)}{\left[\left(\frac{s}{.935}\right)^2+\frac{2(.505)}{.935}s+1\right]\left(\frac{s}{a}+1\right)^2}$$

$$\frac{\Delta\alpha}{\Delta\delta_c} = w_\alpha = \frac{K_\alpha(\tau_\alpha s+1)}{\left[\left(\frac{s}{\omega_n}\right)^2+\frac{2\zeta}{\omega_n}s+1\right]}\;\frac{1}{\left(\frac{s}{a}+1\right)^2} = \frac{-2.4\left(\frac{s}{19.4}+1\right)}{\left[\left(\frac{s}{.935}\right)^2+\frac{2(.505)}{.935}s+1\right]\left(\frac{s}{a}+1\right)^2}$$

Thus,

$$\Delta\delta_{c_o} = \frac{1}{\left[1+\frac{q_1}{r}w_{\dot{\theta}}\bar{w}_{\dot{\theta}}+\frac{q_2}{r}w_\alpha\bar{w}_\alpha\right]^+}\left[\frac{\frac{q_1}{r}w_a w_{\dot{\theta}}+\frac{q_2}{r}w_b\bar{w}_\alpha}{\left[1+\frac{q_1}{r}w_{\dot{\theta}}\bar{w}_{\dot{\theta}}+\frac{q_2}{r}w_\alpha\bar{w}_\alpha\right]^-}\right]_+ \tag{8-16}$$

and the first problem facing the analyst is to investigate the expression

$$\frac{1}{1+\frac{q_1}{r}w_{\dot{\theta}}\bar{w}_{\dot{\theta}}+\frac{q_2}{r}w_\alpha\bar{w}_\alpha} \tag{8-17}$$

Consider the block diagram of Figure 33.

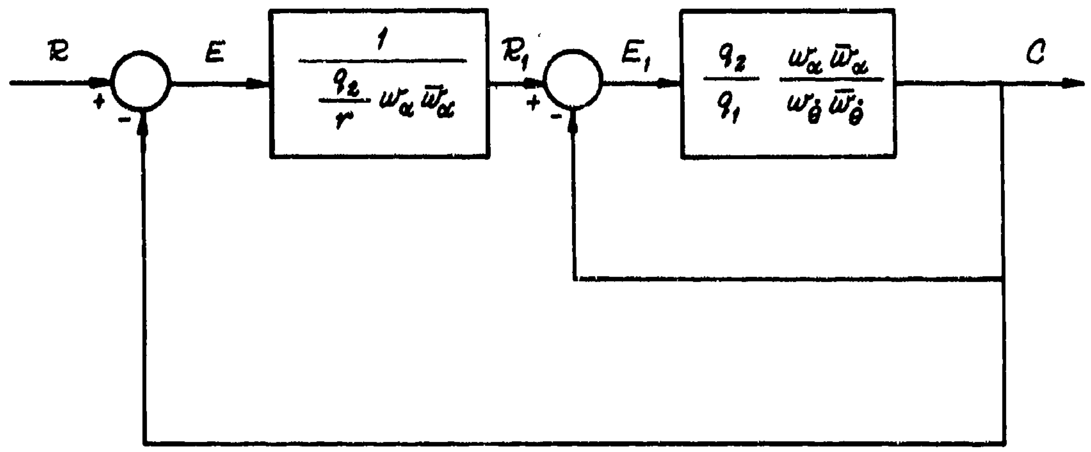


Figure 33. Unity Feedback Form for Design Problem

Using block diagram algebra, one readily finds

$$\frac{C}{R} = \frac{1}{1 + \frac{\hat{q}_1}{r} w_{\dot\theta} \bar{w}_{\dot\theta} + \frac{\hat{q}_2}{r} w_{\alpha} \bar{w}_{\alpha}} \tag{8-18}$$

Thus two successive uses of the open- to closed-loop Bode plot technique will yield the desired result. The obvious advantage in this formulation of the problem is that the poles of the system cancel out in the first step and one can leave the task of investigating the effect of actuator poles, etc., until the second stage of the procedure.

Before proceeding with the analysis, it is in order to point out that we are involved only in a two-parameter investigation in terms of q_1/r and q_2/r. Moreover, the optimal procedure assures us that our closed-loop system will be stable for whatever values of q_1/r and q_2/r are selected. A conventional trial and error design for the system presents a situation which can be very complex indeed. For example, it is conceivable that in executing a conventional design, the analyst might be forced to manipulate as many as eight parameters (since each transfer function is of the fourth order) and moreover, he must be continually checking to see if his closed-loop system is stable.

We now investigate the first stage of the procedure.

$$\frac{w_\alpha}{w_{\dot\theta}} = \frac{K_\alpha}{K_{\dot\theta}} \frac{(\tau_\alpha s+1)}{(\tau_{\dot\theta} s+1)} = \frac{-2.4}{-1.175} \frac{\left(\frac{s}{19.4}+1\right)}{\left(\frac{s}{.495}+1\right)}$$

$$= 2.04 \frac{\left(\frac{s}{19.4}+1\right)}{\left(\frac{s}{.495}+1\right)} \tag{8-19}$$

Therefore

$$\frac{q_2}{q_1} \frac{w_\alpha \bar{w}_\alpha}{w_{\dot\theta} \bar{w}_{\dot\theta}} = \frac{q_2}{q_1}(4.16) \frac{\left(\frac{s}{19.4}+1\right)\left(\frac{-s}{19.4}+1\right)}{\left(\frac{s}{.495}+1\right)\left(\frac{-s}{.495}+1\right)}$$

$$= \frac{q_2}{q_1}(4.16) \left. \frac{\left|\frac{s}{19.4}+1\right|^2}{\left|\frac{s}{.495}+1\right|^2} \right|_{s=j\omega} \tag{8-20}$$

This plot is given in Figure 34 for $q_2/q_1 = 2.4$ (note: $4.16 \approx 12$ db, therefore, $q_2/q_1 \times 4.16 \approx 20$ db).

One now either invokes the open- to closed-loop approximation or uses the 0° line on a Nichols chart to find the closed-loop C/R_1 (see Figure 29).

This closed-loop frequency response is also shown on Figure 34. One easily deduces the analytic form of $C/\mathcal{R}_1$ directly from the plot. Note the response breaks from 0 to -2 and then back to 0.* Fitting a straight line to the exact closed-loop response gives a new break frequency of $\omega = 1.5$. Since the actual response is down 6 db at this frequency, and since only a break of -2 is involved, we conclude

$$\frac{C}{\mathcal{R}_1} \approx \frac{\left(\frac{s}{19.4}+1\right)\left(\frac{-s}{19.4}+1\right)}{\left(\frac{s}{1.5}+1\right)\left(\frac{-s}{1.5}+1\right)}$$

Of course, this procedure is simple to carry out for other values of q_2/q_1. Figure 35 gives the exact Bode plots for $C/\mathcal{R}_1$ as a function of q_2/q_1. From Figure 35, one arrives at the following approximations:

1) For $$\frac{q_2}{q_1} = \frac{q_2/r}{q_1/r} \geq 500$$

$$\frac{C}{\mathcal{R}_1} \approx 1$$

Therefore

$$\frac{C}{\mathcal{R}} = \frac{1}{1+\frac{q_1}{r}\,w_{\dot\theta}\,\bar{w}_{\dot\theta}+\frac{q_2}{r}\,w_{\alpha}\,\bar{w}_{\alpha}} \approx \frac{1}{1+\frac{q_2}{r}\,w_{\alpha}\,\bar{w}_{\alpha}}$$

2) For $$\frac{q_2}{q_1} = \frac{q_2/r}{q_1/r} \leq .01$$

$$\frac{C}{\mathcal{R}_1} \approx \frac{q_2}{q_1}\,\frac{w_{\alpha}\,\bar{w}_{\alpha}}{w_{\dot\theta}\,w_{\dot\theta}}$$

Therefore

$$\frac{C}{\mathcal{R}} = \frac{1}{1+\frac{q_1}{r}\,w_{\dot\theta}\,\bar{w}_{\dot\theta}+\frac{q_2}{r}\,w_{\alpha}\,\bar{w}_{\alpha}} \approx \frac{1}{1+\frac{q_1}{r}\,w_{\dot\theta}\,\bar{w}_{\dot\theta}}$$

We now move on to the second stage of the investigation. In light of the results of the first stage, it seems reasonable to break up the analysis in three parts:

A. $\frac{q_2}{q_1} \leq .01$

That is, investigate the closed-loop poles as a function of q_1/r, using the expression:

$$\frac{1}{1+\frac{q_1}{r}\,w_{\dot\theta}\,\bar{w}_{\dot\theta}}$$

---

* 0 represents 0 db/decade, -2 represents -40 db/decade, etc.

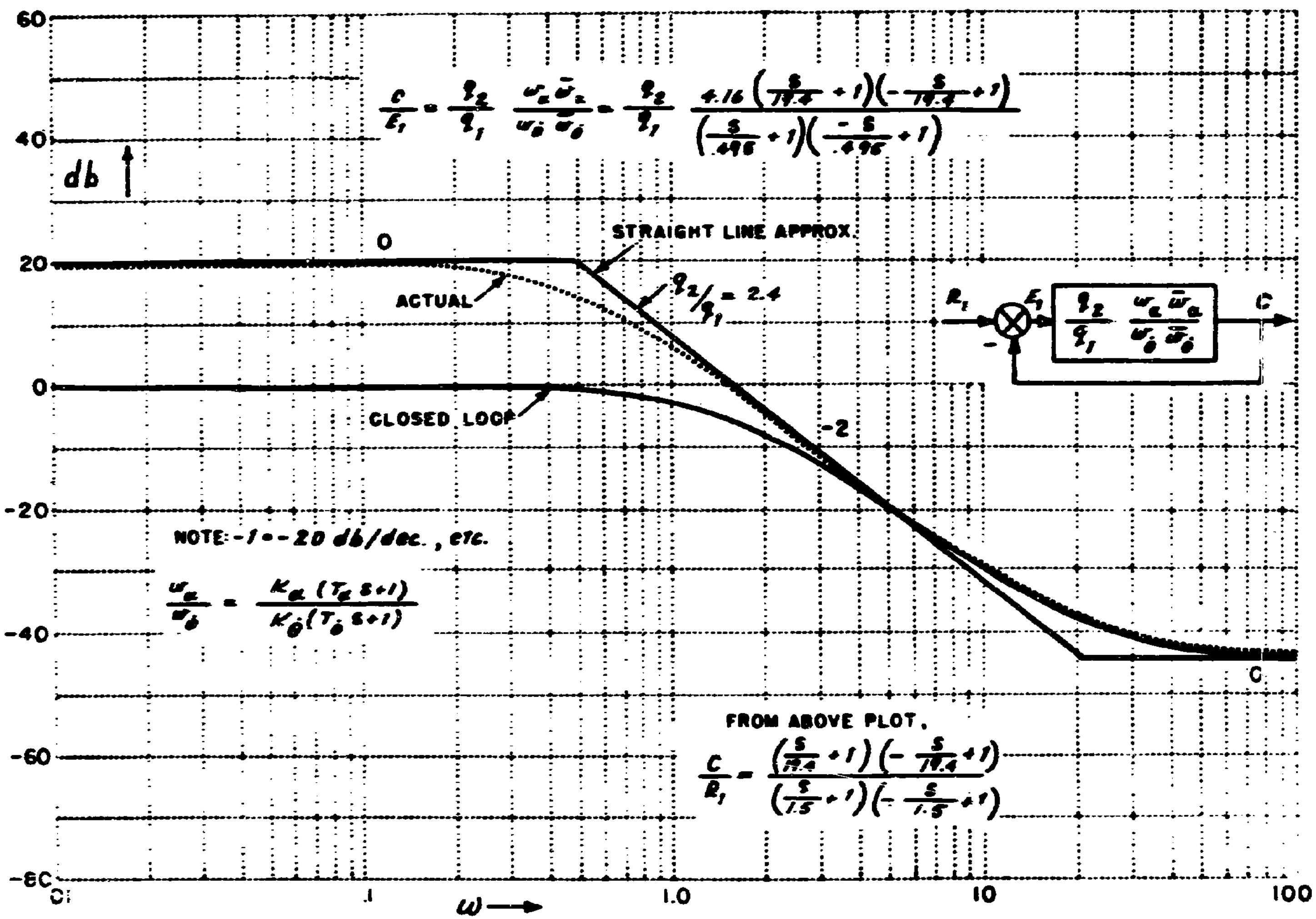


Figure 34. Bode Plot of First Stage, $q_2/q_1 = 2.4$

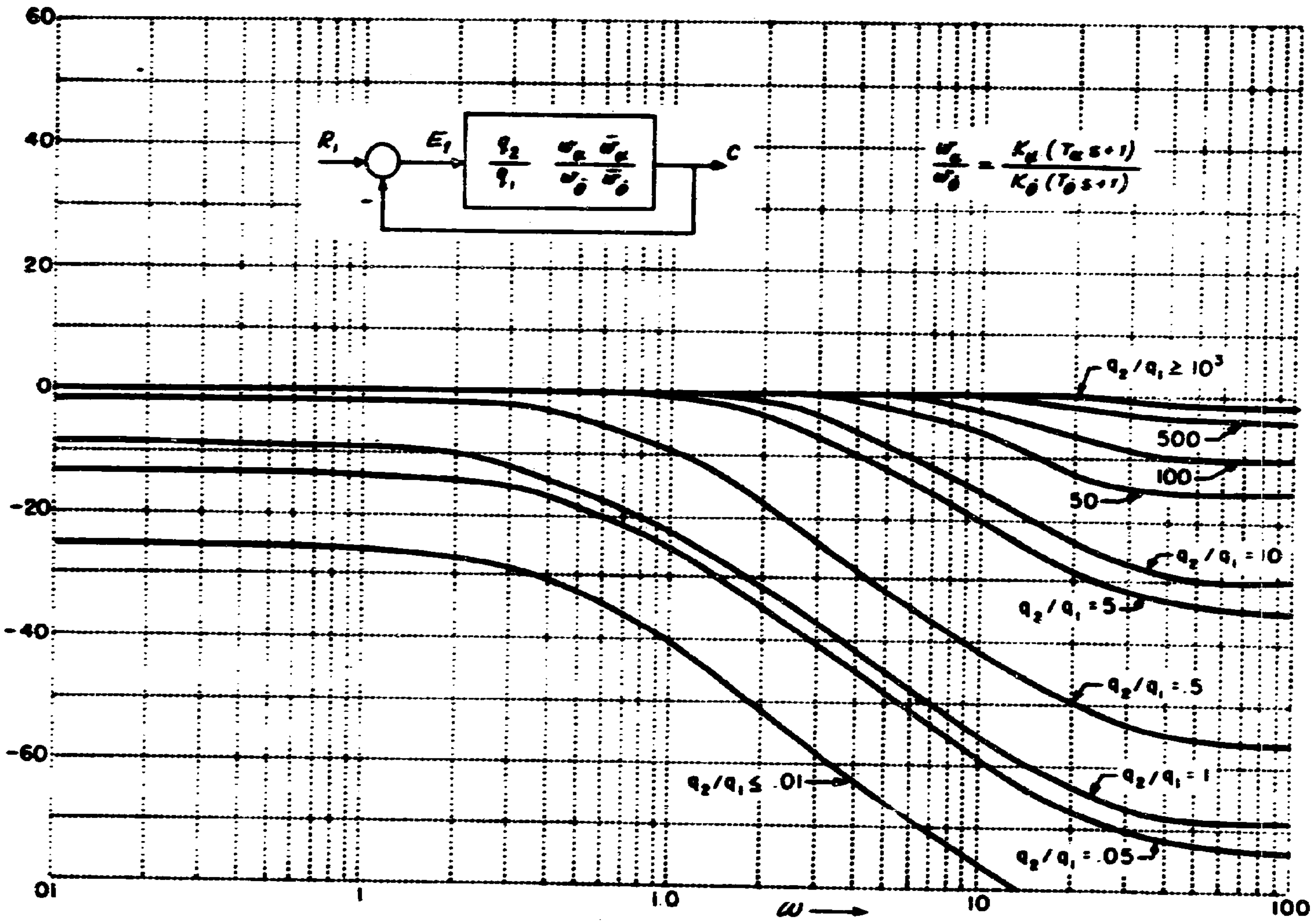


Figure 35. Bode Plot of First Stage, q_2/q_1 Variable

B. $\frac{q_2}{q_1} \geq 500$

That is, investigate the closed-loop poles as a function of q_2/r, using the expression:

$$\frac{1}{1 + \frac{q_2}{r} w_\alpha \bar{w}_\alpha}$$

C. Investigate a typical intermediate situation, such as $q_2 = q_1$.

Therefore,

$$\frac{q_2}{q_1} = 1.0$$

Part A

When

$$\frac{q_2}{q_1} < .01 \; (i.e., \; q_1 \gg q_2)$$

$$\frac{C}{R} = \frac{1}{1 + \frac{q_1}{r} w_\delta \bar{w}_\delta}$$

and

$$\frac{C}{E} = \frac{1}{\frac{q_1}{r} w_\delta \bar{w}_\delta}$$

where

$$\frac{1}{w_\delta} = \frac{\left[\left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n} s + 1\right] \left(\frac{s}{a} + 1\right)^2}{K_\delta (\tau_\delta s + 1)}$$

$$= \frac{\left[\left(\frac{s}{.935}\right)^2 + \frac{2(.605)}{.935} s + 1\right]\left(\frac{s}{a} + 1\right)^2}{-1.175 \, (s/.49 + 1)} \qquad (8\text{-}21)$$

Let the root of the actuator, along with q_1/r, be a parameter of the investigation. Most boost actuators used in aircraft have natural frequencies between .5 and 15 cps.

We can now investigate the behavior of C/R by using straight line approximations. In Figure 36, the open-loop response and the straight line approximations to the closed-loop, for different values of q_1/r, have been plotted for a = 2, 13, and 50 rad/sec. For any given value of q_1/r the approximate expression for

$$\frac{1}{1 + \frac{q_1}{r} w_\delta \bar{w}_\delta}$$

can be read by inspection. For example, when $q_1/r = 10^4$ and a = 2 rad/sec

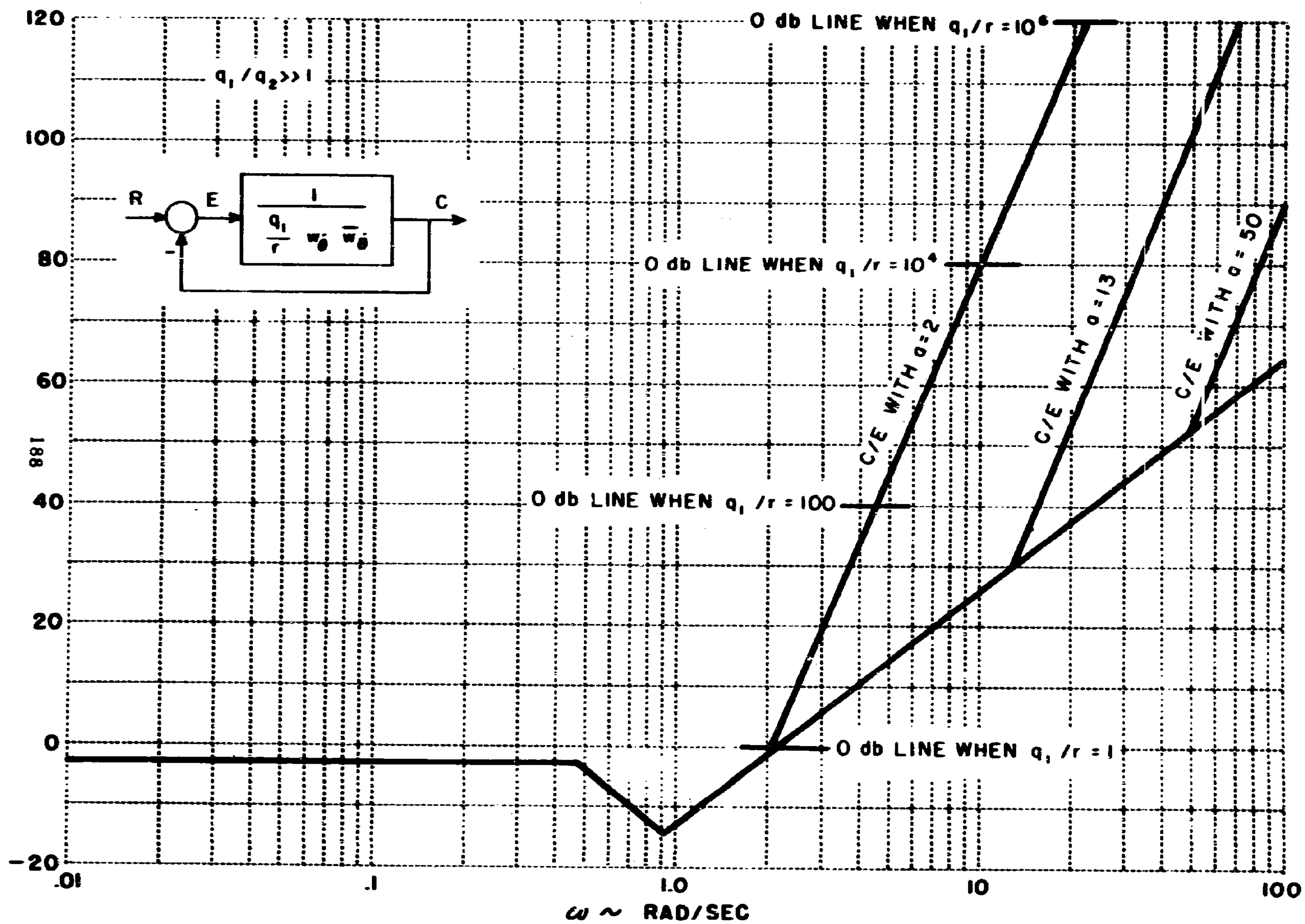


Figure 36. Closed-Loop Poles as a Function of q_1/r

$$\frac{C}{R} = \frac{(7.24\times10^{-5})\left[\left(\frac{s}{.935}\right)^2+\frac{2(.505)}{.935}s+1\right]\left[\left(\frac{s}{.935}\right)^2-\frac{2(.505)}{.935}s+1\right]\left(\frac{s}{2}+1\right)^2\left(-\frac{s}{2}+1\right)^2}{\left(\frac{s}{.49}+1\right)\left(-\frac{s}{.49}+1\right)\underbrace{\left[\left(\frac{s}{10}\right)^2+\frac{2(.5)}{10}s+1\right]\left(\frac{s}{10}+1\right)}\left[\left(\frac{s}{10}\right)^2-\frac{2(.5)}{10}s+1\right]\left(-\frac{s}{10}+1\right)} \tag{8-22}$$

Note: a three-pole Butterworth configuration

When $q_1/r < .01$,

$$\frac{C}{R} \doteq 1.0 \tag{8-23}$$

When $q_1/r = 100$,

$$\frac{C}{R} \doteq \frac{(7.24\times10^{-3})\left[\left(\frac{s}{.935}\right)^2+\frac{2(.505)}{.935}s+1\right]\left[\left(\frac{s}{.935}\right)^2-\frac{2(.505)}{.935}s+1\right]\left(\frac{s}{2}+1\right)^2\left(-\frac{s}{2}+1\right)^2}{\left(\frac{s}{.49}+1\right)\left(-\frac{s}{.49}+1\right)\left[\left(\frac{s}{4.5}\right)^2+\frac{2(.5)}{4.5}s+1\right]\left(\frac{s}{4.5}+1\right)\left[\left(\frac{s}{4.5}\right)^2-\frac{2(.5)}{4.5}s+1\right]\left(-\frac{s}{4.5}+1\right)} \tag{8-24}$$

Of course, our interest is only in the left-half plane poles of Equation 8-24.

From Figure 32, it can be seen that the $\Delta\alpha$ output is always

$$w_\alpha \Delta\delta_C$$

When the optimal control is found, the required feedback gains can be computed. Closing the loops in Figure 32 using the values of the optimal gains will force a $\Delta\alpha$ output which is exactly

$$w_\alpha \Delta\delta_{C_0} \tag{8-25}$$

From Equation 8-16 the form of $\Delta\delta_{C_0}$ can be shown to be

$$\Delta\delta_{C_0} = \frac{D\xi}{\Delta\Gamma} \tag{8-26}$$

where Δ = roots of root square locus

Γ = poles of the model

D = open-loop poles

$\xi(s)$ is a polynomial which results from the partial fraction expansion of the bracketed term in Equation 8-16.

Thus the closed-loop α output is

$$\Delta\alpha_0 = w_\alpha\Delta\delta_{C_0} = \frac{-2.4\left(\frac{s}{19.4}+1\right)}{D}\cdot\frac{D\xi}{\Delta\Gamma}$$

$$= \frac{\left(\frac{s}{19.4}+1\right)\xi(s)}{\underbrace{\left(\frac{s}{.49}+1\right)\left[\left(\frac{s}{10}\right)^2+\frac{2(.5)}{10}s+1\right]\left(\frac{s}{10}+1\right)}_{\Delta}\underbrace{\left[\left(\frac{s}{4.61}\right)^2+\frac{2(.6)}{4.61}s+1\right]}_{\Gamma}} \tag{8-27}$$

for $q_1/r = 10^4$, a = 2 rad/sec and $\xi_c(s) = -2.4\,\xi(s)$. Experience indicates that $\xi_1(s)$ will not contain the factor $(s/.49 + 1)$. Hence the optimal $\Delta\alpha$ response will be dominated by the pole at s = -.49.

On the other hand, the optimal $\Delta\dot{\theta}$ response is

$$\omega_{\dot{\theta}}\,\Delta\delta_{c_0}$$

and will have closed-loop dynamics for $q_1/r = 10^4$ and a = 2 rad/sec proportional to:

$$\Delta\dot{\theta}_c \approx \frac{\cancel{\left(\frac{s}{.495}+1\right)}}{\underbrace{\cancel{\left(\frac{s}{.495}+1\right)}\left[\left(\frac{s}{10}\right)^2+\frac{2(.5)}{10}s+1\right]\left(\frac{s}{10}+1\right)}_{\Delta}\underbrace{\left[\left(\frac{s}{4.51}\right)^2+\frac{2(.6)}{4.51}s+1\right]}_{\Gamma}} \qquad (8\text{-}28)$$

The model poles will dominate the response, if we consider a separation of slightly greater than one octave between the model and plant poles as being sufficient.

Thus we see that our chances of obtaining good model following in $\Delta\dot{\theta}$ are good for the case where $q_1 >> q_2$, but the $\Delta\alpha$ response will in all probability be very poor. The value of the actuator root does not materially affect this result since the dominant root in the $\Delta\alpha$ response is at such a low break frequency (s = -.49). We now proceed to Part B.

Part B

The search must be continued if our interest lies in obtaining a good α response as well as a good $\Delta\dot{\theta}$ response. Consider now the case where $q_2/q_1 \geq 500$. In this event

$$\frac{1}{1+\frac{q_1}{r}\omega_{\dot{\theta}}\bar{\omega}_{\dot{\theta}}+\frac{q_2}{r}\omega_{\alpha}\bar{\omega}_{\alpha}} \approx \frac{1}{1+\frac{q_2}{r}\omega_{\alpha}\bar{\omega}_{\alpha}} \qquad (8\text{-}29)$$

Refer to Figure 37 for the Bode plot associated with this situation. The parameters for this investigation are q_2/r and a, the actuator root. From Figure 37, it is apparent that the open-loop zero at s = -19.4 will become the dominant pole introduced by the plant for the following approximate values of q_2/r:

$q_2/r \geq 10^9$ for a = 2 (extrapolated from plot)

$q_2/r \geq 10^6$ for a = 13

$q_2/r \geq 10^5$ for a = 50

Since the higher values of q_2/r imply higher feedback gains, it is obvious that a high penalty will be paid if a poor actuator is incorporated into the system. If it is assumed that a = 50 and $q_2/r \geq 10^5$, then the design objective of good model following can be achieved in both $\Delta\alpha$ and $\Delta\dot{\theta}$. To illustrate, assume $q_2/r = 10^6$ and a = 50.

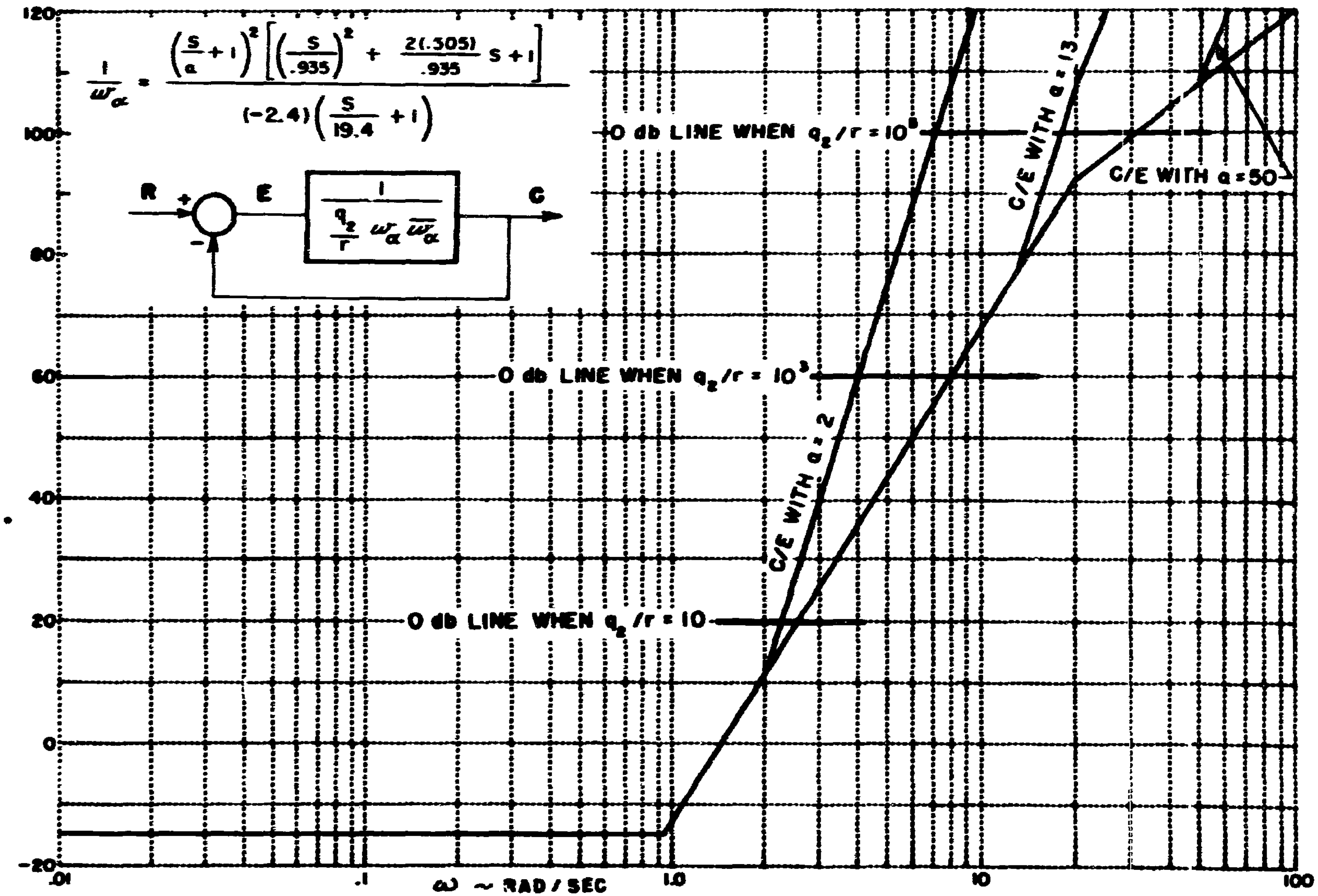


Figure 37. Closed-Loop Roots with q_2/r as a Parameter

$$\Delta\alpha_0 - w_\alpha \Delta\delta_{c_0} \approx \frac{\cancel{\left(\frac{s}{19.4}+1\right)}}{\cancel{\left(\frac{s}{19.4}+1\right)}\left(\frac{s}{60}+1\right)\left[\left(\frac{s}{60}\right)^2+\frac{2(.5)}{60}s+1\right]\left[\left(\frac{s}{4.51}\right)^2+\frac{2(.6)}{4.51}s+1\right]} \tag{8-30}$$

plant poles model poles

and

$$\Delta\dot\theta = w_{\dot\theta}\Delta\delta \approx \frac{\left(\frac{s}{.495}+1\right)}{\left(\frac{s}{19.4}+1\right)\left(\frac{s}{60}+1\right)\left[\left(\frac{s}{60}\right)^2+\frac{2(.5)}{60}s+1\right]\left[\left(\frac{s}{4.51}\right)^2+\frac{2(.6)}{4.51}s+1\right]} \tag{8-31}$$

plant poles model poles

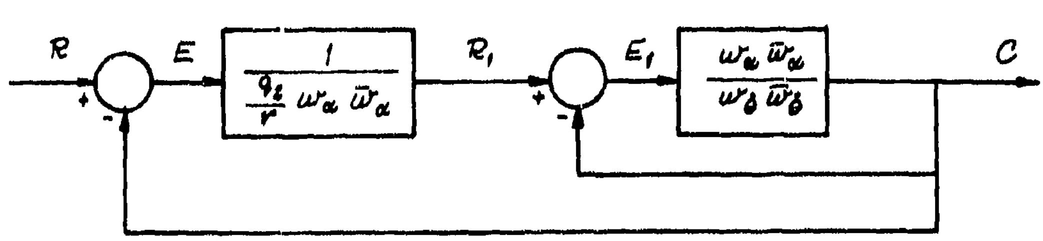
If a separation between plant and model poles of approximately four octaves is acceptable, it is apparent that good model following can probably be achieved in both $\Delta\alpha$ and $\Delta\dot\theta$. However, to verify this, one must find not only the poles of $\Delta\delta_{c_0}$, but the zeros as well. This normally would be the next step in the design procedure, after which the feedback gains would be computed if acceptable $\Delta\dot\theta$ and $\Delta\alpha$ responses were actually found.

At any rate, we have so far discovered that the situation in which $q_1 >> q_2$ does not appear as promising as the case where $q_1 << q_2$. Furthermore, we will probably be interested in investigating more thoroughly the case where a is on the order of 50 rad/sec and q_2/r is on the order of 10^5.

So far, the two extreme cases where $q_1 >> q_2$ and $q_2 >> q_1$ have been investigated. For $q_1 >> q_2$ we found the zero of the $w_{\dot\theta}$ transfer function at s = -.495 became the dominant pole of the optimal control and more than likely would seriously degrade the model-following capability in $\Delta\alpha$. For $q_2 >> q_1$, the zero at s = -19.4 of the w_α transfer function has the potential for becoming the dominant pole of the optimal control and it is felt that good model following in both $\Delta\alpha$ and $\Delta\dot\theta$ can more than likely be achieved. For the intermediate case where $q_2 \approx q_1$, one might suspect that the dominant root will lie between s = -19.4 and s = -.495. This indeed turns out to be the case and we will be forced to conclude that the situation of part B remains the one on which further effort should be expended. To show that the case where $q_2 \approx q_1$ is of not too great interest, we proceed to the analysis of part C.

Part C

It remains to investigate the intermediate situation for which $q_1 \approx q_2$. In this case, the block diagram of Figure 33 reduces to the following block diagram.



Unity Feedback Form for $q_2/q_1 = 1$

An approximate expression for C/R_1 can be found from Figure 35, but it is a simple matter to evaluate it analytically since only first-order breaks are involved. The analytical expression for C/R_1 is found to be

$$\frac{C}{R_1} = \frac{(.81)\left(\frac{s}{19.4}+1\right)\left(-\frac{s}{19.4}+1\right)}{\left(\frac{s}{1.12}+1\right)\left(-\frac{s}{1.12}+1\right)} \tag{8-32}$$

C/E is then of the form

$$\frac{C}{E} = .14\, w \bar{w} \tag{8-33}$$

where

$$w = \frac{\left[\left(\frac{s}{.935}\right)^2 + \frac{2(.505)}{.935}s+1\right]\left[\frac{s}{a}+1\right]^2}{\left(\frac{s}{1.12}+1\right)} \tag{8-34}$$

The Bode plot of Figure 38 summarizes this situation. It is seen that s = -1.12 will always be the dominant root of the optimal control and will not be cancelled out of either the $\Delta\alpha$ or $\Delta\dot{\theta}$ response. This situation defeats the model-following objective entirely. Moreover, it can be seen that other intermediate values of q_2 and q_1 will merely position the dominant root between s = -.49 and s = -19.4. This verifies the assertion made at the close of the analysis of Part B.

At this point, one should return to Part B, solve for the zeros of the optimal control law and find the feedback gains. If these feedback gains are too high to be used in a practical situation, one may either relax the separation requirements between the model poles and plant poles or incorporate another control (e.g., throttle) into the design. We will pursue these considerations no further since our objective was merely to demonstrate how Bode plots might be used as a design aid.

In summary, this brief analysis has shown that the situation in which

$$q_2 \gg q_1$$

$$q_2/r \approx 10^5$$

$$a \approx 50 \text{ rad/sec}$$

is one for which good model following in both α and $\dot{\theta}$ might be expected.

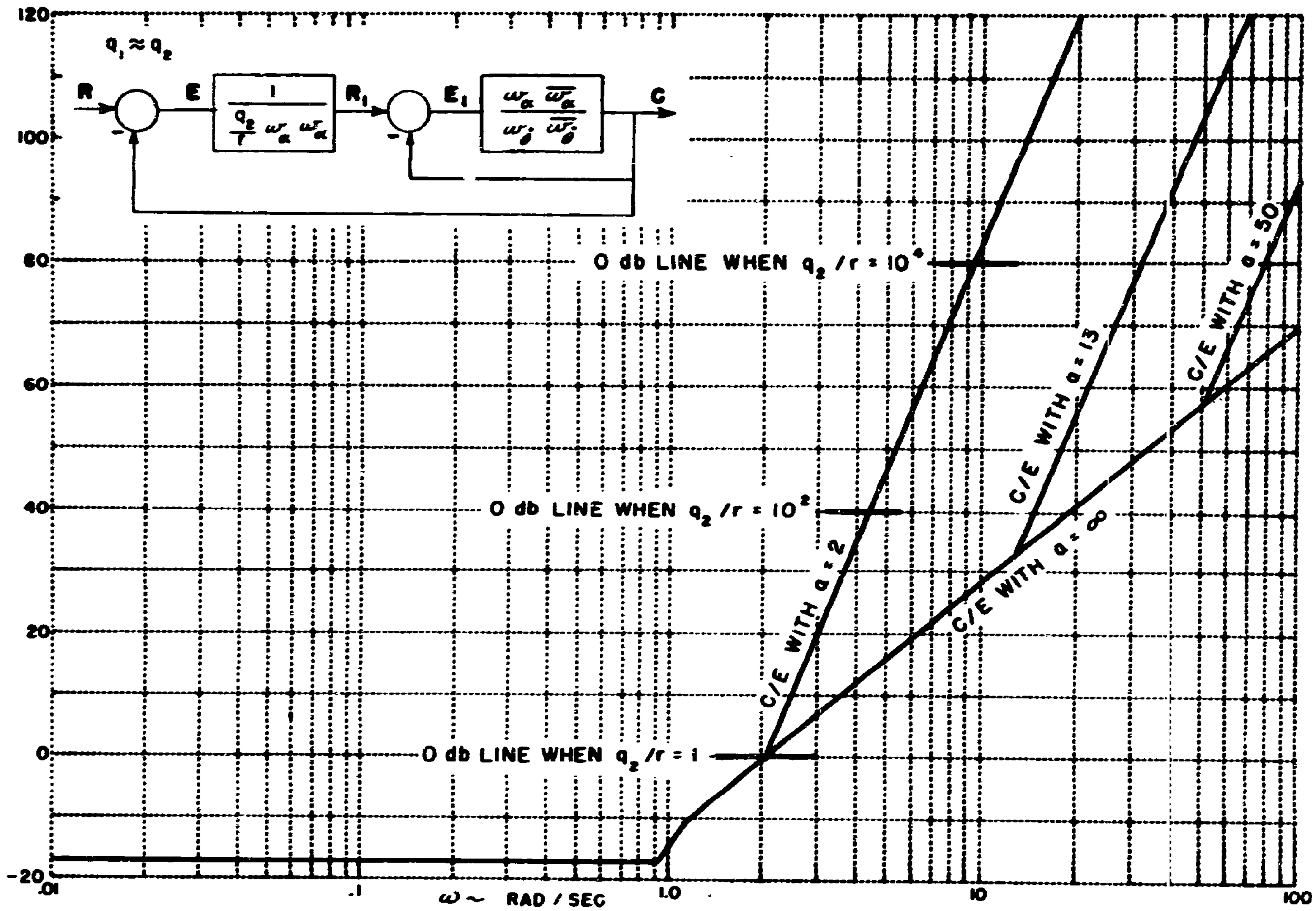


Figure 38. Closed-Loop Roots for $q_1 \approx q_2$ as a Function of q_2/r

SECTION 9

A MULTIVARIABLE EXAMPLE

This section presents a multivariable analysis of a linear optimal control system using the model techniques described previously. The model-in-the-performance-index technique using two control inputs is analyzed using a root square locus. The equivalent Bode plot method is used in conjunction with the model-following idea.

The object to be controlled is a small jet transport and the model is the proposed supersonic transport aircraft. The objective of the analysis is to choose a performance index yielding an optimal control law that, if synthesized, will force the small jet to respond dynamically like the proposed SST within the control capability of the jet transport. The purpose of this section is to demonstrate multivariable analysis techniques, and not necessarily to produce the best design. For this reason, only one set of aerodynamic derivatives has been chosen, and therefore only one flight condition is represented.

9.1 EQUATIONS OF MOTION

The equations of longitudinal motion for the SST and for the small jet transport are assumed to be given by the following:

$$
\begin{aligned}
&\Delta\dot{V} + D_V \Delta V + V_T D_\alpha \Delta\alpha + g\Delta\theta = -V_T D_{\delta_T}\Delta\delta_T \\
&\frac{\Delta\alpha_T}{V_T}\Delta\dot{V} - \frac{1}{V_T} z_V \Delta V + \Delta\dot{\alpha} - z_\alpha \Delta\alpha - \Delta\dot{\theta} - z_\theta \Delta\theta = z_{\delta_e}\Delta\delta_e \qquad (9\text{-}1) \\
&-M_V \Delta V - M_{\dot{\alpha}}\Delta\dot{\alpha} - M_\alpha \Delta\alpha + \Delta\ddot{\theta} - M_q \Delta\dot{\theta} = M_{\delta_e}\Delta\delta_e
\end{aligned}
$$

In first-order form, these equations become

$$
\begin{bmatrix} \Delta\ddot{\theta} \\ \Delta\dot{\theta} \\ \Delta\dot{V} \\ \Delta\dot{\alpha} \end{bmatrix}
=
\begin{bmatrix}
(M_q + M_{\dot{\alpha}}) & \frac{M_{\dot{\alpha}}}{V_T}(g\Delta\alpha_T + V_T z_\theta) & \frac{M_{\dot{\alpha}}}{V_T}(\Delta\alpha_T D_V + z_V) + M_V & M_V + \frac{M_{\dot{\alpha}}\Delta\alpha_T D_\alpha}{V_T} + \frac{M_{\dot{\alpha}} z_\alpha}{V_T} \\
1 & 0 & 0 & 0 \\
0 & -g & -D_V & -V_T D_\alpha \\
1 & \frac{1}{V_T}(g\Delta\alpha_T + V_T z_\theta) & \frac{1}{V_T}(\Delta\alpha_T D_V + z_V) & (\Delta\alpha_T D_\alpha + z_\alpha)
\end{bmatrix}
\begin{bmatrix} \Delta\dot{\theta} \\ \Delta\theta \\ \Delta V \\ \Delta\alpha \end{bmatrix}
$$

$$
+ \begin{bmatrix}
M_{\dot{\alpha}}\Delta\alpha_T D_{\delta_T} & M_{\dot{\alpha}} z_{\delta_e} + M_{\delta_e} \\
0 & 0 \\
-V_T D_{\delta_T} & 0 \\
\Delta\alpha_T D_{\delta_T} & z_{\delta_e}
\end{bmatrix}
\begin{bmatrix} \Delta\delta_T \\ \Delta\delta_e \end{bmatrix} \qquad (9\text{-}2)
$$

These equations can be used to describe both the supersonic transport, which will be used as a model, and the small jet, which will be used as the plant to be controlled. Both of these mathematical model representations are describable by Equation 9-1. The following table lists the aerodynamic derivatives used for a numerical example of the use of models in linear optimal control.

TABLE 3

AERODYNAMIC DERIVATIVES

	V_T (ft/sec)	$\Delta\alpha_T$ (rad)	D_α	D_{δ_T}	D_V	$\mathcal{Z}_V$	$\mathcal{Z}_\theta$
SST	257	.138	-.1190	-.0677	.0263	-.257	-.0173
JET	257	.209	-.0679	-.0329	.0296	-.253	-.0262

	$\mathcal{Z}_\alpha$	$\mathcal{Z}_{\delta_e}$	M_α	$M_{\dot\alpha}$	M_V	M_q	M_{δ_e}
SST	-.842	-.129	-.697	-.514	0	-.771	-.965
JET	-.667	-.0326	-1.75	-.215	.000366	-.536	-2.66

Both of the aircraft are assumed to be in a powered approach flight condition at sea level. It is assumed that the SST is in a light-weight configuration and the jet in a heavy-weight configuration.

Using the derivatives of Table 3 above, the first-order equations of motion are obtained by substitution in Equation 9-2.

The plant (the jet aircraft) becomes:

$$\begin{bmatrix} \Delta\ddot{\theta} \\ \Delta\dot{\theta} \\ \Delta\dot{V} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} -.751 & .0000046 & .000572 & -1.604 \\ 1.000 & 0 & 0 & 0 \\ 0 & -32.2 & -.0296 & 17.45 \\ 1.000 & -.0000214 & -.0009599 & -.681 \end{bmatrix} \begin{bmatrix} \Delta\dot{\theta} \\ \Delta\theta \\ \Delta V \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} .0015 & -2.65 \\ 0 & 0 \\ 8.46 & 0 \\ -.0069 & -.0326 \end{bmatrix} \begin{bmatrix} \Delta\delta_T \\ \Delta\delta_e \end{bmatrix} \tag{9-3}$$

and the model (the SST) becomes:

$$\begin{bmatrix} \Delta\ddot{\theta}_m \\ \Delta\dot{\theta}_m \\ \Delta\dot{V}_m \\ \Delta\dot{\alpha}_m \end{bmatrix} = \begin{bmatrix} -1.285 & .00000442 & .0005067 & -.2558 \\ 1.0 & 0 & 0 & 0 \\ 0 & -32.2 & -.0263 & 30.58 \\ 1.0 & -.0000086 & -.0009859 & -.8584 \end{bmatrix} \begin{bmatrix} \Delta\dot{\theta}_m \\ \Delta\theta_m \\ \Delta V_m \\ \Delta\alpha_m \end{bmatrix} + \begin{bmatrix} .0048 & -.898 \\ 0 & 0 \\ 17.4 & 0 \\ -.00934 & -.129 \end{bmatrix} \begin{bmatrix} \Delta\delta_{T_m} \\ \Delta\delta_{e_m} \end{bmatrix} \tag{9-4}$$

The transfer functions of ΔV and $\Delta\alpha$ for the plant have been obtained, and are given below.

<u>Plant</u>

$$\frac{\Delta V}{\Delta\delta_e}(s) = \frac{904\left(1-\frac{s}{-1.43}\right)\left(1-\frac{s}{69.5}\right)}{\left[\left(\frac{s}{.171}\right)^2+\frac{2(.0571)}{.171}s+1\right]\left[\left(\frac{s}{1.465}\right)^2+\frac{2(.492)}{1.465}s+1\right]}$$

$$\frac{\Delta\alpha}{\Delta\delta_e}(s) = \frac{-1.32\left(1-\frac{s}{-.0146+j.175}\right)\left(1-\frac{s}{-.0146-j.175}\right)\left(1-\frac{s}{-82.3}\right)}{\left[\left(\frac{s}{.171}\right)^2+\frac{2(.0571)}{.171}s+1\right]\left[\left(\frac{s}{1.465}\right)^2+\frac{2(.492)}{1.465}s+1\right]}$$

$$\frac{\Delta V}{\Delta\delta_T}(s) = \frac{-6.23\left(1-\frac{s}{-.719+j1.27}\right)\left(1-\frac{s}{-.719-j1.27}\right)\left(1-\frac{s}{.0215}\right)}{\left[\left(\frac{s}{.171}\right)^2+\frac{2(.0571)}{.171}s+1\right]\left[\left(\frac{s}{1.465}\right)^2+\frac{2(.492)}{1.465}s+1\right]}$$

$$\frac{\Delta\alpha}{\Delta\delta_T}(s) = \frac{-.001305\left(1-\frac{s}{.085}\right)\left(1-\frac{s}{-.085}\right)\left(1-\frac{s}{.688}\right)\left(1-\frac{s}{-4.65}\right)}{\left[\left(\frac{s}{.171}\right)^2+\frac{2(.0571)}{.171}s+1\right]\left[\left(\frac{s}{1.465}\right)^2+\frac{2(.492)}{1.465}s+1\right]}$$

9.2 MODEL IN THE PERFORMANCE INDEX

The problem of including models in a system to obtain a predefined set of closed-loop dynamics is an important concept in linear optimal control. Without a model, the closed-loop optimal system will adjust towards a Butterworth distribution of the excess closed-loop poles over zeros of the system. If there are no zeros, all the closed-loop frequencies will increase without bound as the elements of the Q matrix are made large with respect to the elements of the R matrix.

The optimal, minimum error square characteristic is desirable, but more is needed, and a model supplies this need. A model defines the points at which the closed-loop poles terminate, yet does not destroy the optimal characteristics of the solution. This means that a system can theoretically be designed to have, within limits, any closed-loop regulator dynamics definable by a model. As the elements on the Q matrix become large with respect to the elements of the R matrix, the closed-loop regulator poles do not increase without bound; they can be made to terminate at the model poles. The result is a closed-loop system approaching the system defined by the model.

The supersonic transport will be used as the model for this multivariable

example and the jet aircraft is to be used as the plant to be controlled. The small jet is assumed to have two control inputs; elevator deflection changes and variations of thrust about the nominal. As demonstrated in Section 3, the model and the plant cannot be matched exactly because there are four state variables and only two available controllers. Therefore, for the purposes of this example, it was decided to match the two short period poles of the small jet to those of the SST.

As discussed in Section 3, the model is defined:

$$\dot{\eta} = L\eta \tag{9-5}$$

and the plant is

$$\dot{x} = Fx + Gu \qquad y = Hx \tag{9-6}$$

The performance index is

$$2V = \int_0^\infty \left[(\dot{y} - Ly)'Q(\dot{y} - Ly) + u'Ru\right] dt \tag{9-7}$$

From Equation 3-80, it was found that the expression for the root square locus, and therefore the expression for the closed-loop poles of the optimal system, is given by:

$$\left| I + R^{-1}G'[-Is-F']^{-1}H'[-Is-L']Q[Is-L]H[Is-F]^{-1}G \right| = 0 \tag{9-8}$$

In order to match the short period poles of the small jet to those of the SST, it will be necessary to express this requirement exactly in the performance index. The way to accomplish this is to transform the L and the F matrices into a modal matrix form. In other words, the model matrix becomes

$$L = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix} = \Lambda_L \tag{9-9}$$

where L_1, L_2, L_3 and L_4 are the eigenvalues of the model matrix. In a similar manner, it is necessary to transform the equations of motion of the plant such that the transformed system matrix is a modal matrix whose diagonal entries are the open-loop roots of the plant. In order to obtain this modal matrix, it is necessary to find a transformation T such that an orthogonal state vector, $\tilde{x}$, is obtained,

$$x = T\tilde{x} \tag{9-10}$$

Substituting into the plant equations of motion (9-6), the new equations become

$$\dot{\tilde{x}} = T^{-1}FT\tilde{x} + T^{-1}Gu \qquad \tilde{y} = HT\tilde{x}$$
$$= \Lambda_F \tilde{x} + \tilde{G}u \qquad = \tilde{H}\tilde{x} \tag{9-11}$$

where Λ_F is the modal matrix

$$\Lambda_F = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{9-12}$$

With the use of these newly defined quantities, a performance index can be formulated that will exactly express the desire to match closed-loop poles of the plant to the eigenvalues of the model. The performance index becomes

$$2V = \int_0^\infty [(\dot{\tilde{y}} - \Lambda_L \tilde{y})'Q(\dot{\tilde{y}} - \Lambda_L \tilde{y}) + u'Ru]dt \tag{9-13}$$

With this formulation of the problem, the output matrix $\tilde{H}$ can be chosen to select only those roots of the plant that are to be matched to the eigenvalues of the model.

If the model modal matrix is arranged such that the eigenvalues L_3 and L_4 are the short period poles of the model and λ_3 and λ_4 are the open-loop short period poles of the plant, the output matrix $\tilde{H}$ may be chosen:

$$\tilde{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9-14}$$

With this formulation of $\tilde{H}$, the performance index will contain only the short period dynamics of the plant and the model to the exclusion of the phugoid dynamics.

In terms of the transformed quantities, the root square locus can be expressed as follows:

$$\left| I + R^{-1}\tilde{G}'[-Is - \Lambda_F]^{-1}\tilde{H}'[-Is - \Lambda_L]Q[Is - \Lambda_L]\tilde{H}[Is - \Lambda_F]^{-1}\tilde{G} \right| = 0 \tag{9-15}$$

Every entry of Equation 9-15 is known with the exception of Q and R. Because the root square locus is a function of the ratio of the elements of Q and R, the R matrix may be chosen to be the unit matrix

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9-16}$$

The matrix Q is chosen to be a diagonal matrix

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \tag{9-17}$$

When substituting the constituent matrices into the expression for the root square locus in order to perform a computation, it has been found convenient to start with the Q matrix, alternately pre-multiplying and post-multiplying until the entire product $\tilde{G}'[-Is-\Lambda_R]^{-1}\tilde{H}'[-Is-\Lambda_L]Q[Is-\Lambda_L]\tilde{H}[Is-\Lambda_R]^{-1}\tilde{G}$ is obtained.

First, compute $[-Is-\Lambda_L]Q[Is-\Lambda_L]$

$$[-Is-\Lambda_L]Q[Is-\Lambda_L] = \begin{bmatrix} q_1(-s^2+L_1^2) & 0 & 0 & 0 \\ 0 & q_2(-s^2+L_2^2) & 0 & 0 \\ 0 & 0 & q_3(-s^2+L_3^2) & 0 \\ 0 & 0 & 0 & q_4(-s^2+L_4^2) \end{bmatrix} \tag{9-18}$$

and $\tilde{H}'(-Is-\Lambda_L)Q(Is-\Lambda_L)\tilde{H}$ becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_3(-s^2+L_3^2) & 0 \\ 0 & 0 & 0 & q_4(-s^2+L_4^2) \end{bmatrix} \tag{9-19}$$

A review of the computation at this point already reveals a few interesting facts. The elements $q_1(-s^2+L_1^2)$ and $q_2(-s^2+L_2^2)$ are not included in $\tilde{H}'[-Is-\Lambda_L]Q\,[Is-\Lambda_L]\tilde{H}$ and will therefore not appear at all in the expression for the root square locus. These terms could have been included if $\tilde{H}$ had been a unit matrix.

If the above computation procedure is continued, the total root square locus expression will be obtained:

$$
0 = \begin{vmatrix} 1 + \left. \begin{matrix} q_3 \tilde{g}_{31}^2 \Lambda_3 \bar{\Lambda}_3 (-s^2 - L_3^2) \\ + q_4 \tilde{g}_{41}^2 \Lambda_4 \bar{\Lambda}_4 (-s^2 + L_4^2) \end{matrix} \middle/ D\bar{D} \right. & \left. \begin{matrix} q_3 \tilde{g}_{31} \tilde{g}_{32} \Lambda_3 \bar{\Lambda}_3 (-s^2 - L_3^2) \\ + q_4 \tilde{g}_{41} \tilde{g}_{42} \Lambda_4 \bar{\Lambda}_4 (-s^2 - L_4^2) \end{matrix} \middle/ D\bar{D} \right. \\ \left. \begin{matrix} q_3 \tilde{g}_{31} \tilde{g}_{32} \Lambda_3 \bar{\Lambda}_3 (-s^2 - L_3^2) \\ + q_4 \tilde{g}_{41} \tilde{g}_{42} \Lambda_4 \bar{\Lambda}_4 (-s^2 - L_4^2) \end{matrix} \middle/ D\bar{D} \right. & 1 + \left. \begin{matrix} q_3 \tilde{g}_{32}^2 \Lambda_3 \bar{\Lambda}_3 (-s^2 - L_3^2) \\ + q_4 \tilde{g}_{42}^2 \Lambda_4 \bar{\Lambda}_4 (-s^2 - L_4^2) \end{matrix} \middle/ D\bar{D} \right. \end{vmatrix} \tag{9-20}
$$

$$D\bar{D} = (-s^2 + \lambda_1^2)(-s^2 + \lambda_2^2)(-s^2 + \lambda_3^2)(-s^2 + \lambda_4^2)$$

where $\Lambda_3 \bar{\Lambda}_3 = (-s^2 + \lambda_1^2)(-s^2 + \lambda_2^2)(-s^2 + \lambda_4^2)$

$$\Lambda_4 \bar{\Lambda}_4 = (-s^2 + \lambda_1^2)(-s^2 + \lambda_2^2)(-s^2 + \lambda_3^2)$$

This root square locus expression can be simplified to the following

$$
0 = 1 + \frac{q_3(-s^2 - L_3^2)(\tilde{g}_{31}^2 + \tilde{g}_{32}^2)}{(-s^2 + \lambda_3^2)} + \frac{q_4(-s^2 - L_4^2)(\tilde{g}_{41}^2 + \tilde{g}_{42}^2)}{(-s^2 + \lambda_4^2)} + \frac{q_3 q_4 (\tilde{g}_{31} \tilde{g}_{42} - \tilde{g}_{32} \tilde{g}_{41})^2 (-s^2 - L_3^2)(-s^2 - L_4^2)}{(-s^2 + \lambda_3^2)(-s^2 + \lambda_4^2)} \tag{9-21}
$$

The last term of this root square locus expression is the most important. It defines the end points of the locus. These end points are $(-s^2 - L_3^2)$ and $(-s^2 - L_4^2)$, the two chosen short period roots of the model. As the product $q_3 q_4 \to \infty$, the short period roots of the small jet will become equal to the model short period. Notice that the jet phugoid roots do not appear in the root square locus expression at all. The two eigenvalues λ_1 and λ_2 have been made unobservable by the collineatory transformation on the original state variables, and they are completely excluded from the performance index. The transformed state variables associated with λ_1 and λ_2 do not appear in the feedback control law.

Numerical Example

The small jet and the SST equations of motion of Section 9.1 will be used to show a numerical example of the modal technique. To simplify the problem and still show a non-trivial example, it was decided to delete the small terms f_{12}, f_{13}, f_{42} and f_{43} of the original F matrix of Equation 9-3. These terms contribute primarily to the phugoid motions of the aircraft, such that the deletion will result in a phugoid pole at the origin and a negative real phugoid root. The purpose of this example will be to demonstrate the use

of two controllers to exactly control two roots without affecting the remaining two roots. The short period roots are the ones that will be changed, so the phugoid root locations will have little effect on the problem result.

After deleting f_{12} , f_{13} , f_{42} and f_{43} from F, the matrix is of the form

$$F = \begin{bmatrix} f_{11} & 0 & 0 & f_{14} \\ 1 & 0 & 0 & 0 \\ 0 & f_{32} & f_{33} & f_{34} \\ 1 & 0 & 0 & f_{44} \end{bmatrix} \tag{9-22}$$

and the SST model matrix, with the same approximations, becomes

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 & \ell_{14} \\ 1 & 0 & 0 & 0 \\ 0 & \ell_{32} & \ell_{33} & \ell_{34} \\ 1 & 0 & 0 & \ell_{44} \end{bmatrix} \tag{9-23}$$

The matrix [Is - F] becomes

$$Is - F = \begin{bmatrix} s-f_{11} & 0 & 0 & -f_{14} \\ -1 & s & 0 & 0 \\ 0 & -f_{32} & s-f_{33} & -f_{34} \\ -1 & 0 & 0 & s-f_{44} \end{bmatrix} \tag{9-24}$$

The open-loop characteristic equation of the small jet is given by

$$|Is-F| = s(s-f_{33})\left[s^2-(f_{11}+f_{44})s+f_{11}f_{44}-f_{14}\right] = 0 \tag{9-25}$$

The open-loop roots of the small jet are obtained from the solutions of Equation 9-25. These roots are:

$$\left.\lambda_1 = 0 \qquad \lambda_2 = f_{33} \quad \right\} \text{Roots associated with the phugoid}$$

$$\left.\lambda_3, \lambda_4 = \frac{f_{11}+f_{44}}{2} \pm \frac{1}{2}\sqrt{(f_{11}+f_{44})^2+4f_{14}-4f_{11}f_{44}} \quad \right\} \text{Short period roots}$$

Correspondingly, the open-loop poles of the SST are:

$$\left.\begin{array}{ll} L_1 = 0 & L_2 = f_{33} \end{array}\right\} \text{Phugoid}$$

$$\left. L_3 , L_4 = \frac{f_{11}+f_{44}}{2} \pm \frac{1}{2}\sqrt{(f_{11}+f_{44})^2 + 4f_{14} - 4f_{11}f_{44}} \right\} \text{Short period}$$

Before the actual root square locus expression can be constructed, the collineatory transformation T must be obtained and the equations of motion of the small jet must be transformed into the set

$$\begin{aligned} \dot{\tilde{x}} &= T^{-1}FT\tilde{x} + T^{-1}Gu & \qquad \tilde{y} &= HT\tilde{x} \\ &= \Lambda_F\,\tilde{x} + \tilde{G}u & &= \tilde{H}\tilde{x} \end{aligned}$$

The transformation T is an n x n matrix whose columns T_i satisfy the equations (Reference 13).

$$\left[I\lambda_i - F\right]\left[T_i\right] = 0 \tag{9-26}$$

The column matrix T_i is called a modal column. It is a column of the T matrix associated with the eigenvalue λ_i of the modal matrix Λ_F . The determinant $|I\lambda_i - F|$ = 0 by definition, so Equation 9-26 has an infinite number of solutions, including the trivial solution $T_i = 0$. Any non-trivial solution of Equation 9-26 may be chosen for the i[th] column of the T matrix. To illustrate, consider the root $\lambda_1 = 0$. Equation 9-26 becomes

$$\begin{bmatrix} -f_{11} & 0 & 0 & -f_{14} \\ -1 & 0 & 0 & 0 \\ 0 & -f_{32} & -f_{33} & -f_{34} \\ -1 & 0 & 0 & -f_{44} \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \\ t_{41} \end{bmatrix} = 0$$

or

$$\begin{aligned} -f_{11}t_{11} - f_{14}t_{41} &= 0 \\ -t_{11} &= 0 \\ -f_{32}t_{21} - f_{33}t_{31} - f_{34}t_{41} &= 0 \\ -t_{11} - f_{44}t_{41} &= 0 \end{aligned} \tag{9-27}$$

Equations 9-27 have a non-trivial solution

$$\begin{aligned} t_{11} &= 0 \\ t_{21} &= f_{33} \\ t_{31} &= -f_{32} \\ t_{41} &= 0 \end{aligned} \tag{9-28}$$

Proceeding in a similar manner with the other three eigenvalues of $|Is - F|$, the required orthogonal transformation T can be

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ f_{33} & 0 & -\frac{1}{\lambda_3} & \frac{1}{\lambda_4} \\ -f_{32} & 1 & -\left[\frac{\lambda_3 f_{34} + f_{32}(\lambda_3 - f_{44})}{\lambda_3(\lambda_3 - f_{44})(\lambda_3 - f_{33})}\right] & \left[\frac{\lambda_4 f_{34} + f_{32}(\lambda_4 - f_{44})}{\lambda_4(\lambda_4 - f_{44})(\lambda_4 - f_{33})}\right] \\ 0 & 0 & -\frac{1}{(\lambda_3 - f_{44})} & \frac{1}{(\lambda_4 - f_{44})} \end{bmatrix} \tag{9-29}$$

It has been proven that the transformation T always exists for distinct λ_i. If the roots are repeated roots, a distinct improbability among aircraft, the transformation may or may not exist.

In order to complete the transformation of the original equations of motion of the plant to the orthogonal form, the matrix $\tilde{G} = T^{-1}G$ must be obtained. The inverse of T is

$$T^{-1} = \frac{T^{adj}}{|T|} = \frac{1}{|T|} \begin{bmatrix} T_{11} & -T_{21} & T_{31} & -T_{41} \\ -T_{12} & T_{22} & -T_{32} & T_{42} \\ T_{13} & -T_{23} & T_{33} & -T_{43} \\ -T_{14} & T_{24} & -T_{34} & T_{44} \end{bmatrix} \tag{9-30}$$

where T_{ij} are the minors of $|T|$.

For this particular problem, the first two rows of $\tilde{G}$ do not enter into the root square locus expression of Equation 9-21. It will therefore not be necessary to compute the first two rows of T^{adj}. Since the second row of G is null, the minors T_{23} and T_{24} will not appear in $T^{-1}G$. Therefore, it will be necessary to calculate only T_{13}, T_{33}, T_{43}, T_{14}, T_{34} and T_{44}. It is found that

$$T_{13} = f_{33} t_{44} \qquad T_{14} = f_{33} t_{43}$$

$$T_{33} = 0 \qquad T_{34} = 0 \tag{9-31}$$

$$T_{43} = f_{33} \qquad T_{44} = -f_{33}$$

and

$$|T| = -f_{33}\,(t_{44} + t_{43})$$

The required elements of $\tilde{G}$ are then obtained from

$$\tilde{G}_{1,2} = \frac{-1}{(t_{44} - t_{43})} \begin{bmatrix} t_{44} & -\frac{T_{23}}{f_{33}} & 0 & -1 \\ -t_{43} & \frac{T_{24}}{f_{33}} & 0 & -1 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & 0 \\ g_{41} & g_{42} \end{bmatrix}$$

where $\tilde{G}_{1,2}$ indicates that the first and second rows of $\tilde{G}$ are omitted.

$$\tilde{G}_{1,2} = \frac{-1}{(t_{44}+t_{43})} \begin{bmatrix} g_{11} t_{44} - g_{41} & g_{12} t_{44} - g_{42} \\ -g_{11} t_{43} - g_{41} & -g_{12} t_{43} - g_{42} \end{bmatrix} = \begin{bmatrix} \tilde{g}_{31} & \tilde{g}_{32} \\ \tilde{g}_{41} & \tilde{g}_{42} \end{bmatrix} \tag{9-32}$$

Using the numerical values of f_{ij} from Equations 9-3 and 9-4, it is found that the open-loop short period poles of the small jet and the SST are

$$\lambda_3 = -.7155 + j1.264 \qquad L_3 = -1.072 + j.459$$

$$\lambda_4 = -.7155 - j1.264 \qquad L_4 = -1.072 - j.459$$

Substituting these values of λ_3 and λ_4 into the expression for the T transformation yields

$$T = \begin{bmatrix} 0 & 0 & -1 & 1 \\ -.0296 & 0 & .7155 - j\,1.264 & -.7155 - j\,1.264 \\ 32.2 & 1 & \frac{11.3424 + j\,18.65}{2.2807 - j\,1.3374} & \frac{-13.6286 - j62.778}{2.2807 + j\,1.3374} \\ 0 & 0 & \frac{-1}{-.0355 + j\,1.264} & \frac{1}{-.0355 - j\,1.264} \end{bmatrix} \tag{9-33}$$

The elements of T are complex, and the elements $\tilde{g}_{ij}$ will also be complex. Substituting in 9-32 yields

$$\begin{aligned} \tilde{g}_{31} &= -.00075 + j.00433 \\ \tilde{g}_{41} &= +.00075 + j.00433 \\ \tilde{g}_{32} &= +1.325 + j.0578 \\ \tilde{g}_{42} &= -1.325 + j.0578 \end{aligned} \qquad (9\text{-}34)$$

However, when the elements of $\tilde{g}_{ij}$ are substituted into the root square locus expression, the result does not contain complex quantities.

It was shown that the product $q_3 q_4$ is an important parameter of the root square locus expression. This product determines the rate at which the poles of the plant approach those of the model in the root square locus expression. Because the product $q_3 q_4$ is important, it will be sufficient to let $q_3 = q_4 = q$. With this simplification, the expression for the root square locus becomes:

$$0 = 1 + q \frac{\left[(\tilde{g}_{31}{}^2 + \tilde{g}_{32}{}^2)(-s^2 - L_3{}^2)(-s^2 + \lambda_3{}^2) + (\tilde{g}_{41}{}^2 + \tilde{g}_{42}{}^2)(-s^2 - L_4{}^2)(-s^2 + \lambda_4{}^2)\right]}{(-s^2 + \lambda_3{}^2)(-s^2 + \lambda_4{}^2)}$$

$$+ q^2 \frac{(\tilde{g}_{31}\tilde{g}_{42} - \tilde{g}_{32}\tilde{g}_{41})(-s^2 - L_3{}^2)(s^2 - L_4{}^2)}{(-s^2 + \lambda_3{}^2)(-s^2 + \lambda_4{}^2)} \qquad (9\text{-}35)$$

After substituting for $\tilde{g}_{ij}$, L_k and λ_l , Equation 9-35 can be written:

$$0 = 1 - q \frac{3.504(s^4 + .223 s^2 + .540)}{s^4 + 2.1736 s^2 + 4.455} - q^2 \frac{(.000132)(s^4 - 1.8708 s^2 + 1.8634)}{s^4 + 2.1736 s^2 + 4.455} \qquad (9\text{-}36)$$

The root square locus plot of Equation 9-36 can be easily obtained using one of the many digital programs designed for the purpose. The important observation is that the two plant poles do take on the value of the model roots as q becomes very large. It can be argued that the actual locus itself is relatively unimportant; optimal control has been used as a tool to attain a design goal that can be obtained by conventional techniques. This is true, but the design technique described above has several advantages over a conventional technique:

1. As a parameter is varied to locate one set of roots at a new location, there is guaranteed to be no intermediate parameter value that would result in instability.
2. The two open-loop poles excluded from the root square locus expression are not changed at all from their open-loop values.
3. The method is unique and direct.

It is instructive to construct the locus in order to show how the plant poles are altered to eventually become identical to those of the model. The locus is shown in Figure 39 as a function of q. The figure shows that the

closed-loop poles wander somewhat before finally terminating at the roots of the model.

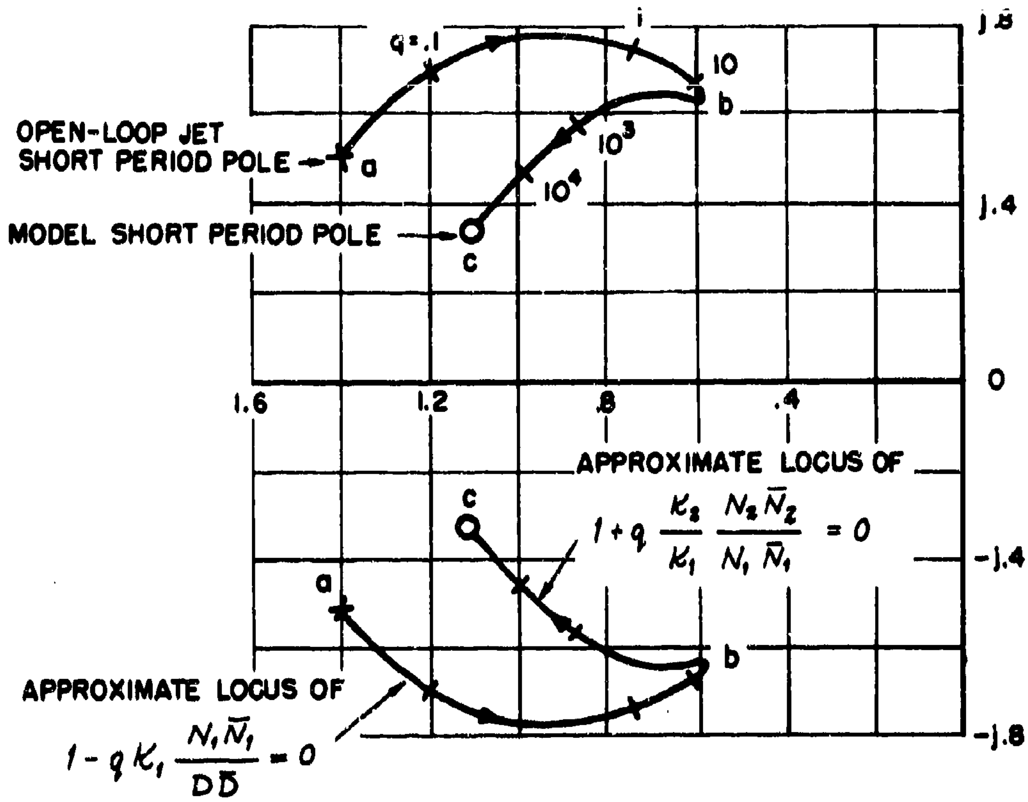


Figure 39. Realizable Part of the Model-in-the-Performance-Index Root Square Locus

The reason for this can be determined from an examination of Equation 9-36. The root square locus expression can be rewritten in the form

$$1 - q K_1 \frac{N_1 \bar{N}_1}{D \bar{D}} \left[1 + q \frac{K_2}{K_1} \frac{N_2 \bar{N}_2}{N_1 \bar{N}_1} \right] = 0 \tag{9-37}$$

For small values of q, $K_1 >> q K_2$, the locus is described very closely by

$$1 - q K_1 \frac{N_1 \bar{N}_1}{D \bar{D}} = 0 \tag{9-38}$$

For large values of q, when $qK_2 \gg K_1$, the locus is approximated by

$$1 - q^2 K_2 \frac{N_2 \bar{N}_2}{D \bar{D}} = 0 \qquad (9\text{-}39)$$

If it is assumed that the low q approximation holds for q smaller than that obtained from

$$\frac{(3.504)(.540)}{4.45} = 10 \times \frac{(.000132)(1.8634)}{4.45} q \qquad (9\text{-}40)$$

or

$$0.425 = 10 \times .0000552\, q$$

then the segment a-b of the locus shown in Figure 39 is approximated closely by Equation 9-38. If it is assumed that the high q approximation is valid for values of q greater than obtained from

$$10 \times .425 = .0000552\, q \qquad (9\text{-}41)$$

then the high q approximation is valid only on that part of the locus very close to the model roots. Because the high q approximation holds only very near the model roots, the intermediate part of the locus, b-c, can be approximated very closely by the expression

$$1 + q \frac{K_2}{K_1} \frac{N_2 \bar{N}_2}{N_1 \bar{N}_1} = 0 \qquad (9\text{-}42)$$

The two segment approximations to the root square locus, namely Equations 9-38 and 9-42 serve to demonstrate that a construction using a root locus plot is possible and not difficult even though a term q^2 appears in the original expression for the root square locus.

It is clear that the procedure of first transforming the system equations of motion into a diagonal form, before analyzing the optimal system, yields a direct method of matching part of a dynamic system to the model dynamics, without affecting the remaining dynamic characteristics. Two controllers were required to accomplish this match, one for each pole of the dynamic system to be altered. It is well known, however, that one controller can normally be used to alter short period aircraft dynamics, but as it turns out, not without affecting the phugoid dynamics of the airframe. This technique, then, has the advantage of directly selecting the dynamics to be altered, and then yielding a feedback configuration that alters the selected dynamics in a reasonable, preselected fashion.

9.3 A COMPLEX MODEL-FOLLOWING DESIGN PROBLEM

A complex model-following design problem will be carried out in this section. We seek an optimal control law that will force a small subsonic jet airplane to have a dynamic response similar to the response predicted by the proposed supersonic transport's equations of motion. That is, the small subsonic transport's equations of motion will yield the F_p and G_p matrices of Equation 7-91 while the supersonic transport will yield the L

matrix. The problem will be limited to one flight condition, since the objective is to demonstrate procedures rather than the execution of a complete design. Specifically, a heavily loaded subsonic jet, flying at sea level and a Mach number of 0.23 will be forced to simulate a lightly loaded supersonic transport flying at sea level and a Mach number of 0.23.

The matrix equations have the form:

$$\begin{bmatrix} \Delta\ddot{\theta} \\ \Delta\dot{\theta} \\ \Delta\dot{V} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} F_p \end{bmatrix} \begin{bmatrix} \Delta\dot{\theta} \\ \Delta\theta \\ \Delta V \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} G_p \end{bmatrix} \begin{bmatrix} \Delta\delta_T \\ \Delta\delta_e \end{bmatrix} \tag{9-43}$$

and

$$\begin{bmatrix} \Delta\ddot{\theta}_m \\ \Delta\dot{\theta}_m \\ \Delta\dot{V}_m \\ \Delta\dot{\alpha}_m \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} \Delta\dot{\theta}_m \\ \Delta\theta_m \\ \Delta V_m \\ \Delta\alpha_m \end{bmatrix} + \begin{bmatrix} G_m \end{bmatrix} \begin{bmatrix} \Delta\delta_{T_m} \\ \Delta\delta_{e_m} \end{bmatrix} \tag{9-44}$$

The G_m matrix is not used in the model-following method (see Section 7.7). For convenience, Equation 7-91 is reproduced below.

$$\begin{aligned} &\left[G_p' \left[-Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1} G_p + R\right] u_0 \\ &+ G_p' \left[Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1} \left[x_p(0) - \left[Is - F_p\right]\left[Is - L\right]^{-1} x_m(0)\right] = z \end{aligned} \tag{9-45}$$

The matrices are given below:

$$F_p = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ 1 & 0 & 0 & 0 \\ 0 & f_{32} & f_{33} & f_{34} \\ 1 & f_{42} & f_{43} & f_{44} \end{bmatrix} \qquad G_p = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & 0 \\ g_{41} & g_{42} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ 1 & 0 & 0 & 0 \\ 0 & l_{32} & l_{33} & l_{34} \\ 1 & l_{42} & l_{43} & l_{44} \end{bmatrix}$$

For the given flight condition, certain matrix entries, which were smaller than the next most significant entry by a factor of approximately 100, were excluded. Thus the matrices used in this design were (these are the same as given in Equations 9-22 and 9-23):

$$F_p = \begin{bmatrix} f_{11} & 0 & 0 & f_{14} \\ 1 & 0 & 0 & 0 \\ 0 & f_{32} & f_{33} & f_{34} \\ 1 & 0 & 0 & f_{44} \end{bmatrix} = \begin{bmatrix} -.751 & 0 & 0 & -1.6 \\ 1 & 0 & 0 & 0 \\ 0 & -32.2 & -.029 & 17.45 \\ 1 & 0 & 0 & -.68 \end{bmatrix} \tag{9-46}$$

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 & \ell_{14} \\ 1 & 0 & 0 & 0 \\ 0 & \ell_{32} & \ell_{33} & \ell_{34} \\ 1 & 0 & 0 & \ell_{44} \end{bmatrix} = \begin{bmatrix} -1.285 & 0 & 0 & -.26 \\ 1 & 0 & 0 & 0 \\ 0 & -32.2 & -.026 & 30.6 \\ 1 & 0 & 0 & -.86 \end{bmatrix} \tag{9-47}$$

$$G_p = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & 0 \\ g_{41} & g_{42} \end{bmatrix} = \begin{bmatrix} 1.5 \times 10^{-3} & -2.65 \\ 0 & 0 \\ 8.5 & 0 \\ -6.9 \times 10^{-3} & -.033 \end{bmatrix} \tag{9-48}$$

The $[Is - F_p]^{-1}$ matrix is given below. The $[Is - L]^{-1}$ is not shown, since it has exactly the same form when the entries are changed from f's to ℓ's.

$$
[Is-F]^{-1} = \frac{\begin{bmatrix} s(s-f_{33})(s-f_{44}) & 0 & 0 & f_{14}\, s(s-f_{33}) \\ (s-f_{33})(s-f_{44}) & \begin{array}{c}(s-f_{33})[s^2-(f_{11}+f_{44})s \\ +f_{11}f_{44}-f_{14}]\end{array} & 0 & f_{14}(s-f_{33}) \\ \begin{array}{c}(f_{32}+f_{34})s \\ -f_{32}f_{44}\end{array} & \begin{array}{c}f_{32}s^2-f_{32}(f_{11}+f_{44})s \\ +f_{32}f_{11}f_{44}-f_{14}f_{32}\end{array} & \begin{array}{c}s[s^2-(f_{11}f_{44})s \\ +f_{11}f_{44}-f_{14}]\end{array} & \begin{array}{c}s^2f_{34}-f_{11}f_{34}s \\ +f_{14}f_{32}\end{array} \\ s(s-f_{33}) & 0 & 0 & s(s-f_{11})(s-f_{33}) \end{bmatrix}}{s(s-f_{33})[s^2-(f_{11}+f_{44})s+f_{11}f_{44}-f_{14}]} \tag{9-49}
$$

The only remaining unknowns are the Q and R matrices. These are determined as soon as one decides which variables are to be controlled and which controls are to do the controlling. In this problem, two controls are available to control four output variables. At this point, we make the subjective decision to control two of the four outputs with the throttle and elevator. The variables ΔV and $\Delta\alpha$ are selected. The desired performance index is:

$$
2V = \int_0^\infty \left\{ q_1(\Delta V_m - \Delta V)^2 + q_2(\Delta\alpha_m - \Delta\alpha)^2 + r_1\,\Delta\delta_T^{\,2} + r_2\,\Delta\delta_e^{\,2} \right\} dt \tag{9-50}
$$

so that

$$
Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \end{bmatrix} \tag{9-51}
$$

and

$$\mathcal{R} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{9-52}$$

As noted in Section 7.7, the unwanted error terms can be excluded from the performance index using either the H or Q matrix. Here it was elected to use the Q matrix. The result which would have been achieved if the H vector had been selected in reality amounts only to rewriting Equation 9-51 in the form:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9-53}$$

The expression

$$G_\rho' [-Is - F_\rho']^{-1} Q [Is - F_\rho]^{-1} G_\rho + \mathcal{R} \tag{9-54}$$

in Equation 9-45 becomes

$$G_\rho' [-Is - F_\rho']^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [Is - F_\rho]^{-1} G_\rho + \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{9-55}$$

Equation 9-55 can then be defined to be

$$W_* \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} W + \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{9-56}$$

where W represents a 2 x 2 matrix of transfer functions. The root square locus is defined by the equation

$$\left| \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} + W_* \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} W \right| = 0 \tag{9-57}$$

The next task is the one of computing W. The routine multiplication of

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \left[Is - F_\rho\right]^{-1} G_\rho \tag{9-58}$$

gives

$$W = \frac{\begin{bmatrix} \begin{matrix} g_{11}\left\{[f_{33}+f_{34}]s - f_{32}f_{44}\right\} \\ + g_{31}\left\{s\left[s^2-(f_{11}+f_{44})s+f_{11}f_{44}-f_{14}\right]\right\} \\ + g_{41}\left\{s^2 f_{34} - f_{11}f_{34}s + f_{14}f_{33}\right\} \end{matrix} & \begin{matrix} g_{12}\left\{[f_{32}+f_{34}]s - f_{32}f_{44}\right\} \\ + g_{42}\left\{s^2 f_{34} - f_{11}f_{34}s + f_{14}f_{32}\right\} \end{matrix} \\ \\ \begin{matrix} g_{11}\, s(s-f_{33}) \\ + g_{41}\, s(s-f_{11})(s-f_{33}) \end{matrix} & \begin{matrix} g_{12}\, s(s-f_{33}) \\ + g_{42}\, s(s-f_{11})(s-f_{33}) \end{matrix} \end{bmatrix}}{s(s-f_{33})\left[s^2-(f_{11}+f_{44})s+f_{11}f_{44}-f_{14}\right]} \tag{9-59}$$

After substituting in numerical values and clearing through to the standard transfer function form, one obtains

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{9-60}$$

where

$$w_{11} = \frac{6.77\left(\frac{s}{.02118}+1\right)\left[\left(\frac{s}{1.495}\right)^2 + \frac{2(.495)}{1.495}s+1\right]}{D} \tag{9-61}$$

$$w_{12} = \frac{920\left(\frac{-s}{68.56}+1\right)\left(\frac{s}{1.427}+1\right)}{D} \tag{9-62}$$

$$w_{21} = \frac{-1.71\times 10^{-3}(s)\left(\frac{s}{.029}+1\right)\left(\frac{s}{.5304}+1\right)}{D} \qquad (9\text{-}63)$$

$$w_{22} = \frac{1.27s\left(\frac{s}{.029}+1\right)\left(\frac{s}{81.054}+1\right)}{D} \qquad (9\text{-}64)$$

and

$$D = s\left(\frac{s}{.029}+1\right)\left[\left(\frac{s}{1.453}\right)^2 + \frac{2(.492)}{1.453}\,s+1\right]$$

We are now ready to proceed with the analysis of the root square locus. The root square locus expression is

$$|R+W_*QW| = |R+W_*QW|$$

$$= \left| \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} + \begin{bmatrix} \bar{w}_{11} & \bar{w}_{21} \\ \bar{w}_{12} & \bar{w}_{22} \end{bmatrix} \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \right| \qquad (9\text{-}65)$$

$$= \begin{vmatrix} r_1+q_1 w_{11}\bar{w}_{11}+q_2 w_{21}\bar{w}_{21} & q_1\bar{w}_{11}w_{12}+q_2\bar{w}_{21}w_{22} \\ q_1 w_{11}\bar{w}_{12}+q_2 w_{21}\bar{w}_{22} & r_2+q_1 w_{12}\bar{w}_{12}+q_2 w_{22}\bar{w}_{22} \end{vmatrix} \qquad (9\text{-}66)$$

$$\begin{aligned} &= r_1 r_2 + r_2 q_1 w_{11}\bar{w}_{11} + r_1 q_1 w_{12}\bar{w}_{12} + r_2 q_2 w_{21}\bar{w}_{21} \\ &\quad + r_1 q_2 w_{22}\bar{w}_{22} + q_1 q_2\left\{(w_{11}w_{22}-w_{12}w_{21})(\bar{w}_{11}\bar{w}_{22}-\bar{w}_{12}\bar{w}_{21})\right\} \\ &= 1+\frac{q_1}{r_1}w_{11}\bar{w}_{11}+\frac{q_1}{r_2}w_{12}\bar{w}_{12}+\frac{q_2}{r_1}w_{21}\bar{w}_{21}+\frac{q_2}{r_2}w_{22}\bar{w}_{22} \\ &\quad +\frac{q_1}{r_1}\frac{q_2}{r_2}\underbrace{\det W}_{Z}\cdot\underbrace{\det W_*}_{\bar{Z}} \end{aligned} \qquad (9\text{-}67)$$

Before proceeding further, an expression for Z = det W must be obtained. After considerable manipulation, one obtains

$$|W| = \frac{g_{31}g_{42}\left[s^2-\left(f_{11}-\frac{g_{12}}{g_{42}}\right)s+f_{33}\left(\frac{g_{11}}{g_{31}}-\frac{g_{12}\,g_{14}}{g_{31}\,g_{42}}\right)\right]}{s(s-f_{33})\left[s^2-(f_{11}+f_{44})s+f_{11}f_{44}-f_{32}\right]} \qquad (9\text{-}68)$$

Substituting in numerical values and placing the result in standard transfer function form gives

$$\bar{Z} = |W| = \frac{9.5\left(\frac{-s}{.026}+1\right)\left(\frac{s}{81.1}+1\right)}{D} \tag{9-69}$$

The roots of Equation 9-67, as a function of q's and r's, can be analyzed using the block diagram of Figure 40, or one may proceed using a similar but more direct method which is described next.

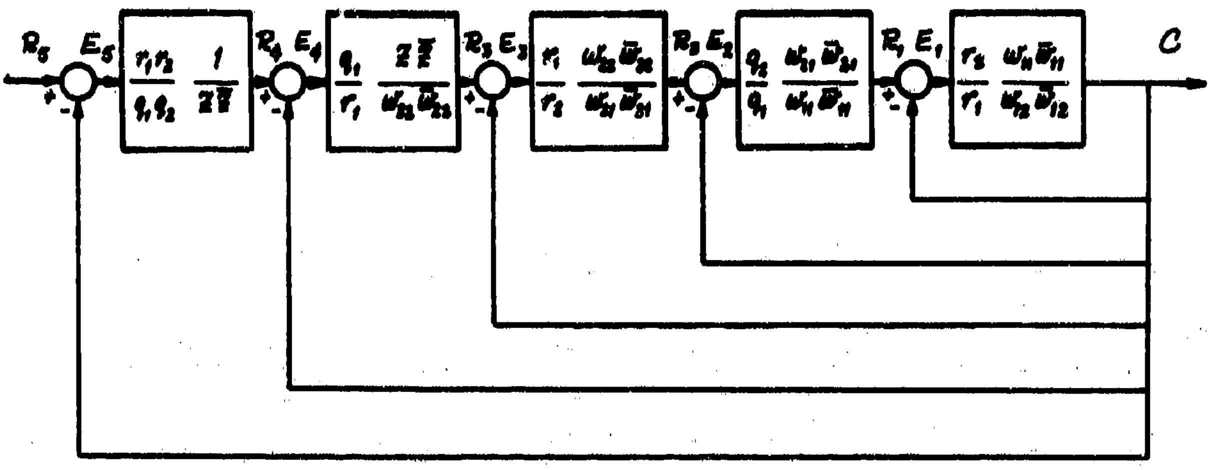


Figure 40. Unity Feedback Block Diagram for Finding Closed-Loop Poles

The objective of the analyses is to obtain a valid approximate expression for the root square locus that will disclose the basic pattern of the closed-loop roots. (This more direct method is possible because the relative importance of the transfer functions of the root square locus change as a function of the q's and r's.)

As a first step, rewrite Equation 9-67 as

$$r_1 r_2 + r_2 q_1 w_{11}\bar{w}_{11}\left(1 + \frac{q_2}{q_1}\,\frac{w_{21}\bar{w}_{21}}{w_{11}\bar{w}_{11}}\right) + r_1 q_1 w_{12}\bar{w}_{12} + r_1 q_2 w_{22}\bar{w}_{22} + q_1 q_2 Z\bar{Z} = 0 \tag{9-70}$$

and investigate the roots of the equation

$$1 + \frac{q_2}{q_1}\,\frac{w_{21}\bar{w}_{21}}{w_{11}\bar{w}_{11}} = 0 \tag{9-71}$$

using a Bode plot and the ratio

$$\frac{w_{21}}{w_{11}} = \frac{(-1.71 \times 10^{-3})\, s \left(\frac{s}{.029}+1\right)\left(\frac{s}{.5304}+1\right)}{6.77\left(\frac{s}{.02118}+1\right)\left[\left(\frac{s}{1.495}\right)^2 + \frac{2(.493)}{1.495}\, s + 1\right]} \tag{9-72}$$

The Bode plot, as a function of q_2/q_1, is given in Figure 41. From this figure, it is seen

$$1 + \frac{q_2}{q_1}\,\frac{w_{21}\bar{w}_{21}}{w_{11}\bar{w}_{11}} \doteq 1.0 \tag{9-73}$$

is a very conservative estimate for

$$0 < \frac{q_2}{q_1} < 10^5 \tag{9-74}$$

Thus the root square locus expression reduces to

$$r_1 r_2 + r_2 q_1 w_{11}\bar{w}_{11} + r_1 q_1 w_{12}\bar{w}_{12} + r_1 q_2 w_{22}\bar{w}_{22} + q_1 q_2 Z\bar{Z} = 0 \tag{9-75}$$

for $0 < q_2/q_1 < 10^5$.

Again, go back to Equation 9-67, this time rewriting it in the form

$$r_1 r_2 + r_2 q_1 w_{11}\bar{w}_{11} + r_1 q_1 w_{12}\bar{w}_{12} + q_2 r_1 w_{22}\bar{w}_{22}\left(1 + \frac{r_2}{r_1}\,\frac{w_{21}\bar{w}_{21}}{w_{22}\bar{w}_{22}}\right) + q_1 q_2 Z\bar{Z} = 0 \tag{9-76}$$

and investigate the expression

$$1 + \frac{r_2}{r_1}\,\frac{w_{21}\bar{w}_{21}}{w_{22}\bar{w}_{22}} \tag{9-77}$$

using the ratio

$$\frac{w_{21}}{w_{22}} = \frac{-1.71 \times 10^{-3}}{1.27}\;\frac{\cancel{s\left(\frac{s}{.029}+1\right)}\left(\frac{s}{.5304}+1\right)}{\cancel{s\left(\frac{s}{.029}+1\right)}\left(\frac{s}{81.054}+1\right)} \tag{9-78}$$

The Bode plot of Figure 42 shows this result. Notice that for a value of $r_2/r_1 = 10$, the open loop is 10 db less than one. Even when $r_2/r_1 = 100$, the frequency range over which the open loop is < 1 is still considerable (i.e., $0 < \omega < 45$ rad/sec). As the second approximation we assume

$$1 + \frac{r_2}{r_1}\,\frac{w_{21}\bar{w}_{21}}{w_{22}\bar{w}_{22}} \approx 1.0 \tag{9-79}$$

for

$$\frac{r_2}{r_1} \leq 10 \tag{9-80}$$

or for $r_2/r_1 \leq 100$, $\omega < 20$ rad/sec.

Equation 9-76 is now identical to Equation 9-75, which was obtained using a different approximation. Thus we have, so far, for the root square locus, the expression

$$r_1 r_2 + r_2 q_1 w_{11} \bar{w}_{11} + r_1 q_1 w_{12} \bar{w}_{12} + r_1 q_2 w_{22} \bar{w}_{22} + q_1 q_2 \; z\bar{z} = 0 \tag{9-81}$$

for either $0 < q_2/q_1 < 10^5$ or $0 < r_2/r_1 < 10$

As a third step, one now writes Equation 9-81 in the form

$$r_1 r_2 + r_2 q_1 w_{11} \bar{w}_{11} + q_2 r_1 w_{22} \bar{w}_{22} + r_1 q_1 w_{12} \bar{w}_{12} \left(1 + \frac{q_2}{r_1} \; \frac{z\bar{z}}{w_{12} \bar{w}_{12}}\right) = 0 \tag{9-82}$$

and investigates

$$1 + \frac{q_2}{r_1} \; \frac{z\bar{z}}{w_{12} \bar{w}_{12}} \tag{9-83}$$

This analysis is shown in Figure 43. We conclude

$$1 + \frac{q_2}{r_1} \; \frac{z\bar{z}}{w_{12} \bar{w}_{12}} \approx 1.0 \tag{9-84}$$

for

$$0 < \frac{q_2}{r_1} \leq 3.0 \tag{9-85}$$

We find that a good approximation for the root square locus is now

$$1 + \frac{q_1}{r_1} \; w_{11} \bar{w}_{11} + \frac{q_1}{r_2} \; w_{12} \bar{w}_{12} + \frac{q_2}{r_2} \; w_{22} \bar{w}_{22} = 0 \tag{9-86}$$

for either $q_2/q_1 \leq 10^5$ or $r_2/r_1 \leq 10$ (9-87)

and when

$$\frac{q_2}{r_1} \leq 3 \tag{9-88}$$

As a fourth step, Equation 9-86 is rewritten as

$$1 + \frac{q_1}{r_1} \; w_{11} \bar{w}_{11} \left(1 + \frac{q_2}{q_1} \; \frac{r_1}{r_2} \; \frac{w_{22} \bar{w}_{22}}{w_{11} \; \bar{w}_{11}}\right) + \frac{q_1}{r_2} \; w_{12} \bar{w}_{12} = 0 \tag{9-89}$$

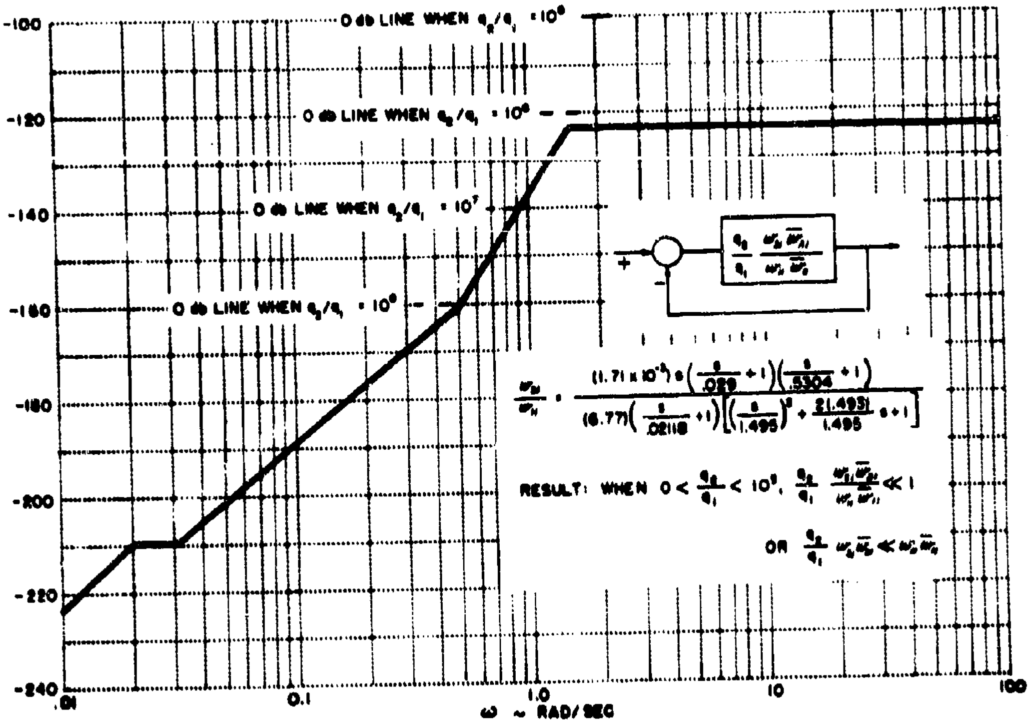


Figure 41. ω_{21}/ω_{22} , q_2/q_1 as the Parameter

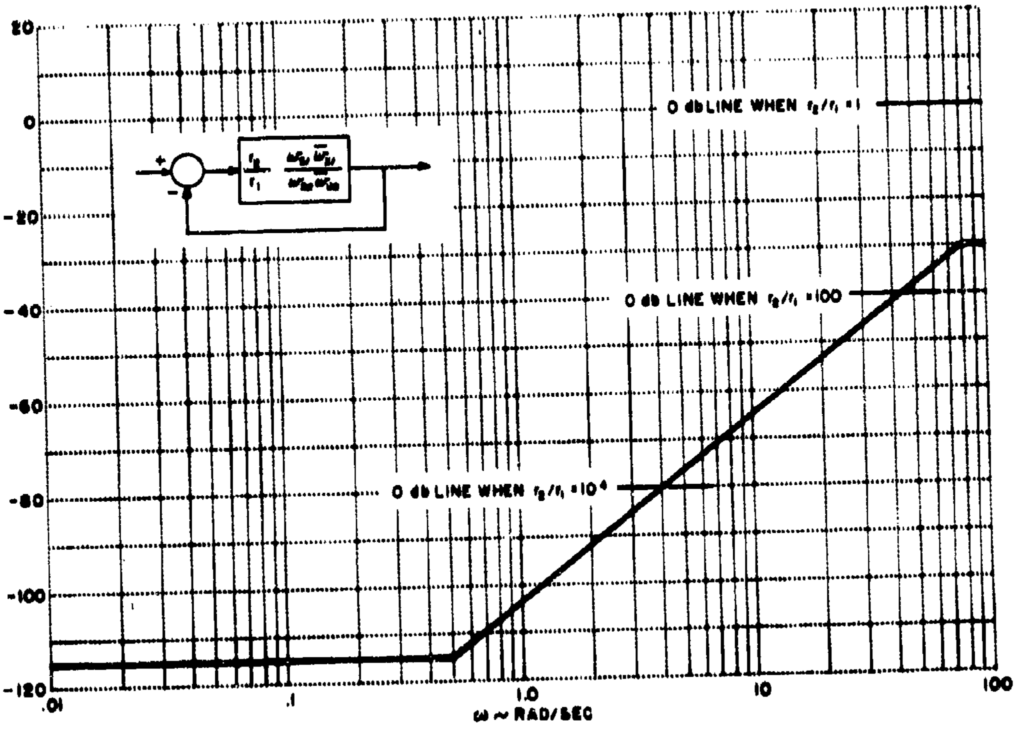


Figure 42. ω_{21}/ω_{22} , r_2/r_1 as the Parameter

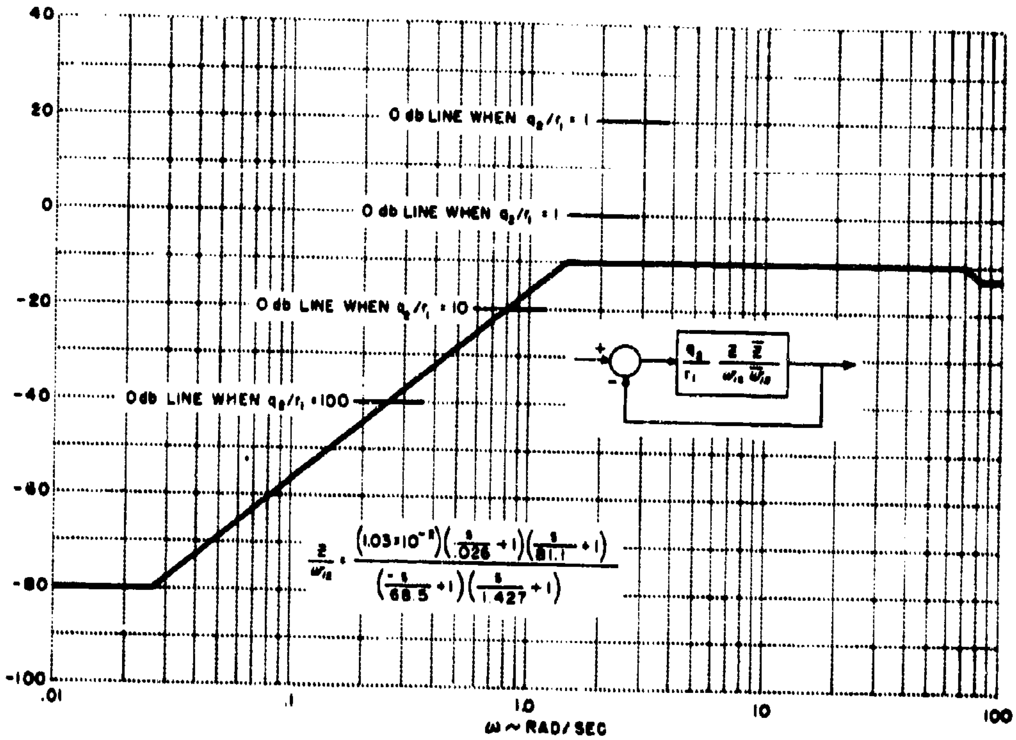


Figure 43. Z/ω_{21} , q_2/r_1 as the Parameter

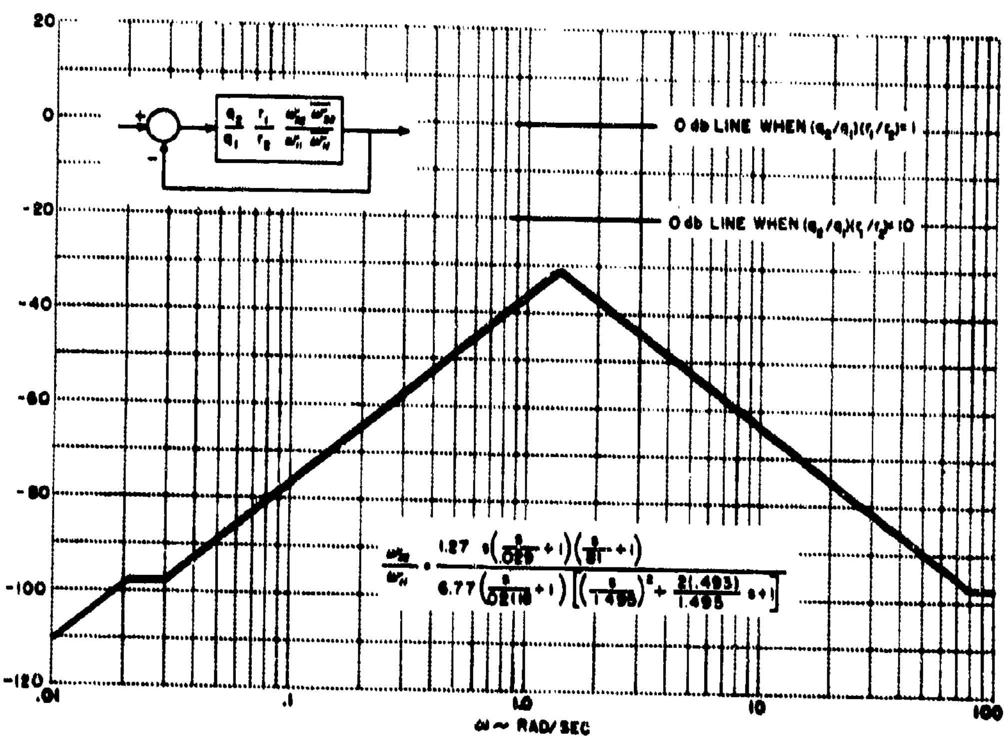


Figure 44. ω_{22}/ω_{11} , r_1/r_2 , q_2/q_1 as the Parameter

The Bode plot for this step is shown in Figure 44. Our approximation here again is a function of frequency, but one can safely conclude

$$1 + \frac{q_2}{q_1}\,\frac{r_1}{r_2}\,\frac{w_{22}\bar{w}_{22}}{w_{11}\bar{w}_{11}} \approx 1.0 \tag{9-90}$$

for

$$\frac{q_2}{q_1} \leq 10\,\frac{r_2}{r_1} \tag{9-91}$$

Even for $q_2/q_1 \leq 100\, r_2/r_1$, the approximation breaks down only over a band of frequencies of approximately $.8 \leq \omega \leq 1.7$.

The root square locus expression now looks like

$$1 + \frac{q_1}{r_1}\,w_{11}\bar{w}_{11} + \frac{q_1}{r_2}\,w_{12}\bar{w}_{12} = 0 \tag{9-92}$$

The conditions under which Equation 9-92 hold are summarized in Figure 45. The cross-hatched area of Figure 45 is the approximate area for which Equation 9-92 remains valid.

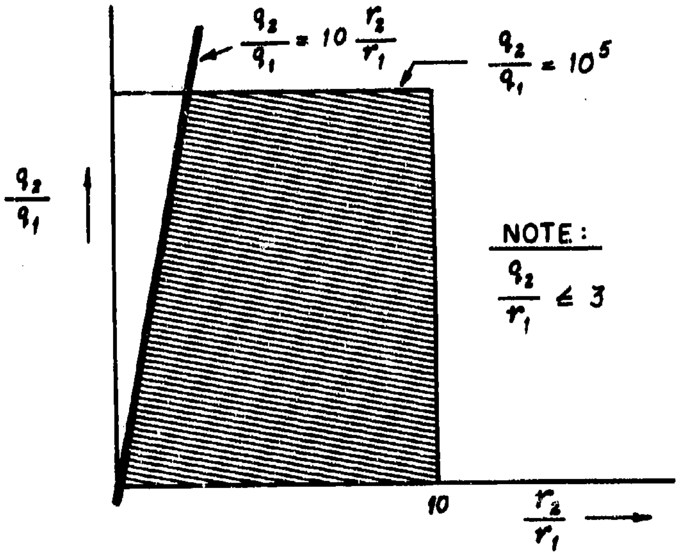


Figure 45. q_2/q_1 Versus r_2/r_1

Equation 9-92 can now be placed in the form

$$1 + \frac{q_1}{r_2}\,w_{12}\bar{w}_{12}\left(1 + \frac{r_2}{r_1}\,\frac{w_{11}\bar{w}_{11}}{w_{12}\bar{w}_{12}}\right) = 0 \tag{9-93}$$

and investigated for

$$0 < \frac{r_2}{r_1} < 10$$

For $r_2/r_1 = 0$, Equation 9-93 is:

$$1 + \frac{q_1}{r_2} \omega_{12} \bar{\omega}_{12} = 0 \tag{9-94}$$

Thus the closed-loop roots can be found using the block diagram:

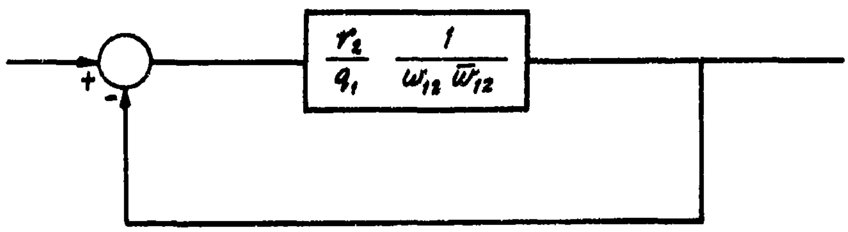


Figure 46. Block Diagram for Approximate Expression

This analysis is shown in Figure 47 as a function of r_2/q_1. The closed-loop poles found using Figure 47 are the roots of Equation 9-94. Since our objective is to move the poles of the plant far enough into the left-hand plane so that they do not interfere with the roots of the model, we should use ratios of $r_2/q_1 < 100$. For example, when

$$r_2/q_1 = .01$$

from Figure 47 one can see that the dominant closed-loop pole becomes the open-loop zero at s = -1.427, while the other three roots form a third-order Butterworth filter pattern with a natural frequency of approximately 7.5 rad per sec.

As r_2/q_1 decreases, the natural frequency of the third-order Butterworth increases, but the dominant root at s = -1.427 remains fixed. Thus the dominant closed-loop root becomes the closest zero to the origin of the ω_{12} transfer function. If we select values of the q's and r's that do not lie too close to the cross-hatched area of Figure 45, one would find that a zero (closest to the origin) of one of the other transfer functions would become the dominant root.

This, of course, is less desirable than the present situation and one is forced to conclude that values of the q's and r's that lie within the cross-hatched area are the ones to investigate for the most desirable closed-loop response. One might now proceed with a systematic investigation of the expression

$$1 + \frac{q_1}{r_2} \omega_{12} \bar{\omega}_{12} + \frac{q_1}{r_1} \omega_{11} \bar{\omega}_{11} = 0 \tag{9-95}$$

for some typical values of q_1, r_1, and r_2 which lie in the cross-hatched area. However, computer results, using the exact root square locus expressions verify that the closed-loop roots are essentially as predicted by the approximate root square locus. It is therefore more interesting to select a set of q's and

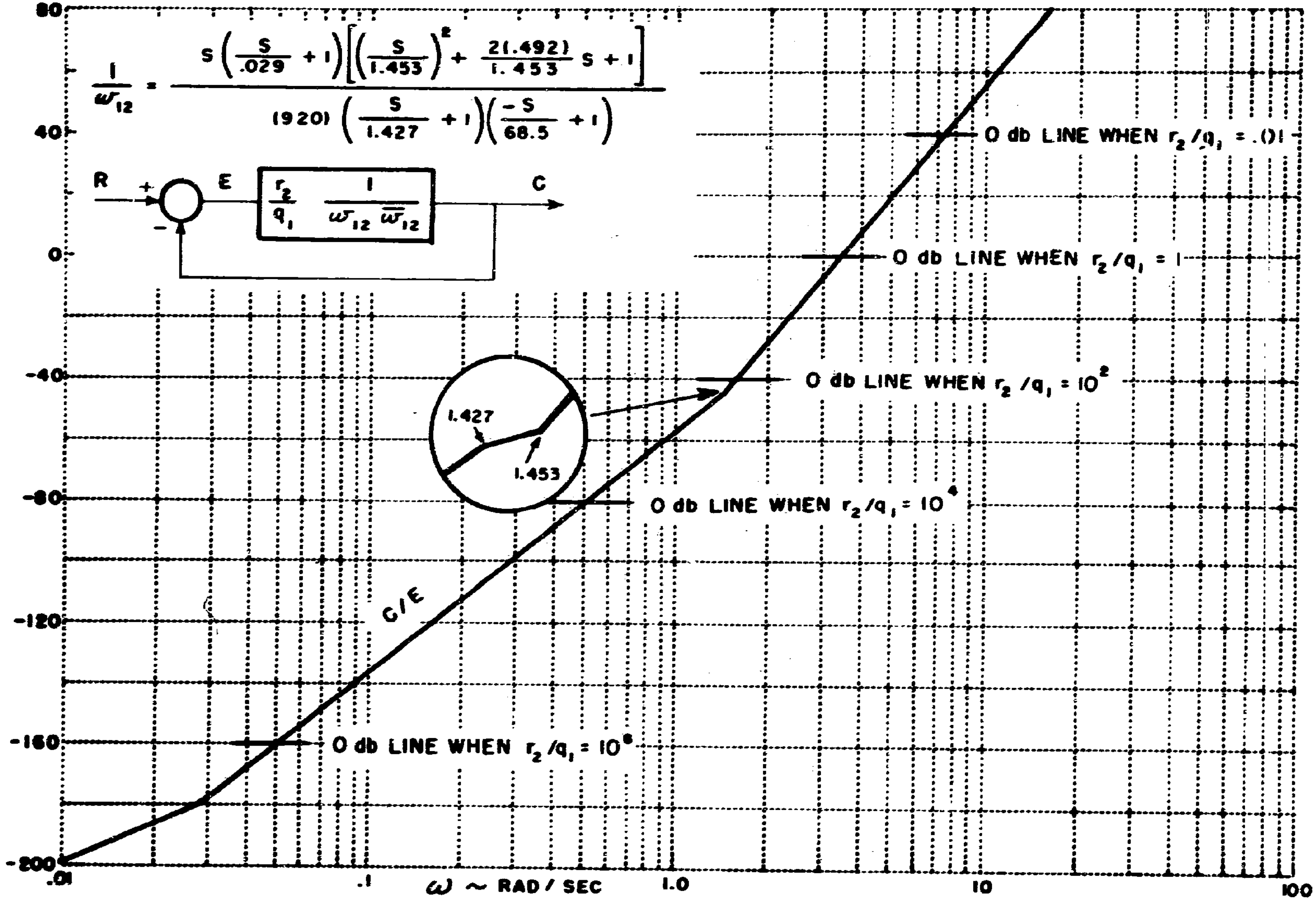


Figure 47. $1/\omega_{12}$, r_2/q_1 as the Parameter

r's which lie slightly outside the boundary and see how the approximate expression compares with the exact one. For the sake of brevity, we will show only one such investigation. The values used were

$$q_1 = q_2 = 100, \quad r_1 = 100, \quad r_2 = 1 \tag{9-96}$$

Therefore, $q_2/q_1 = 1$ and $r_2/r_1 = .01$, a point which lies on the line $q_2/q_1 = 100\, r_2/r_1$.

The block diagram of Figure 48 can be used to investigate this case.

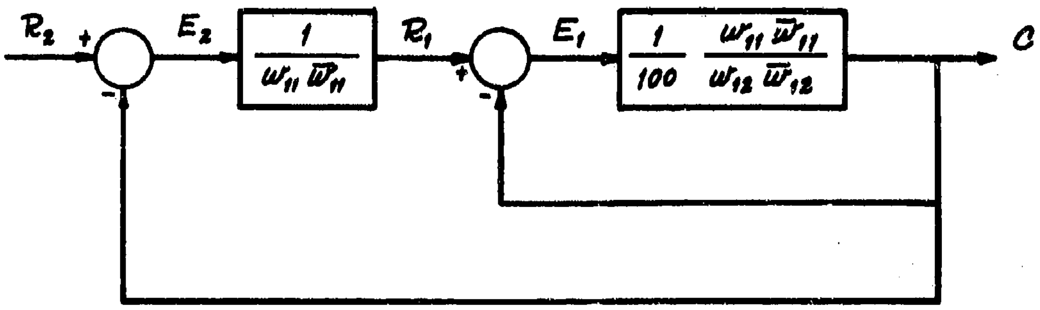


Figure 48. Block Diagram Describing Approximate Root Square Locus

The first stage is shown in Figure 49. The left-half plane components of C/R_1 can be read directly from the Bode plot. The poles and zeros are

$$\frac{\left(\frac{s}{.02118}+1\right)\left[\left(\frac{s}{1.495}\right)^2+\frac{2(.493)}{1.495}s+1\right]}{\left(\frac{s}{1.427}+1\right)\left[\left(\frac{s}{6.6}\right)^2+\frac{2(.707)}{6.6}s+1\right]} \tag{9-97}$$

The loop is closed again and the result read from Figure 50. The closed-loop poles are defined by the polynomial expression

$$\left(\frac{s}{1.427}+1\right)\left[\left(\frac{s}{6.6}\right)^2+\frac{2(.6)}{6.6}s+1\right]\left(\frac{s}{9.5}+1\right) \tag{9-98}$$

The damping factor in Equation 9-98 was found by correcting the straight line approximation in the neighborhood of $\omega = 6.6$ and then looking up the damping factor in any one of a number of standard tables. The optimal control and feedback gains were then found using Kalman's computer program and the given values of q and r.* The polynomial which resulted was

* As of this writing, a computer program does not exist for evaluating the optimal control law and feedback gains using the direct frequency domain approach outlined in Section 7.5. It is estimated that with such a program, the problem described above could be evaluated in 1/100 hour of machine time on an IBM 7044. The existing time domain program required 15 minutes of machine time for this problem.

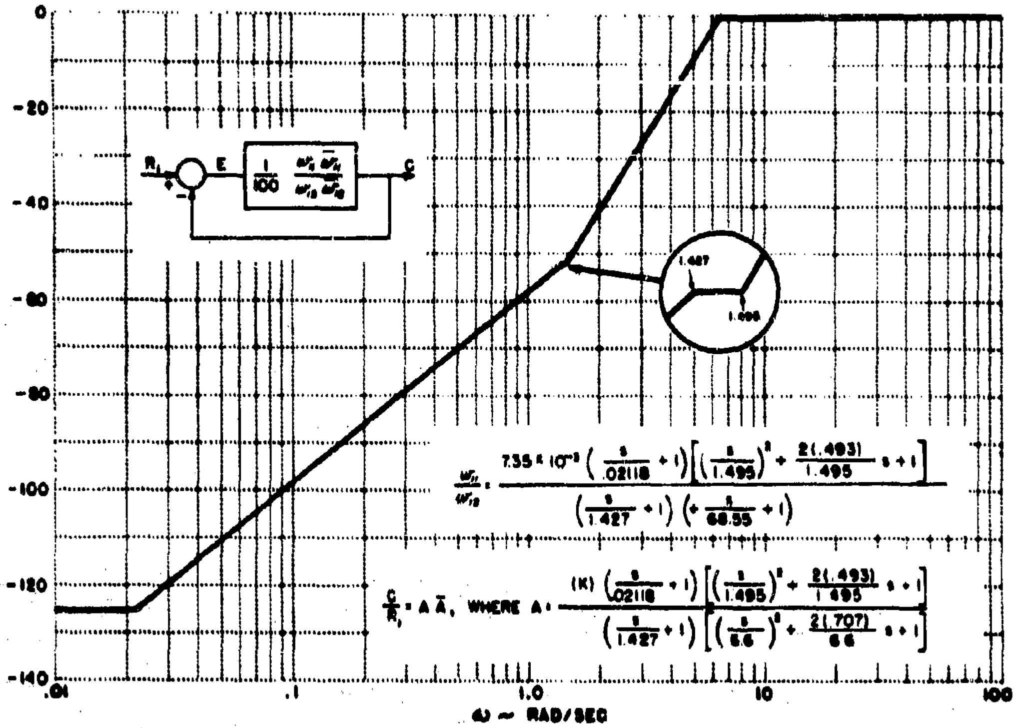


Figure 49. ω_{11}/ω_{12} , $r_3/r_1 = 1/100$

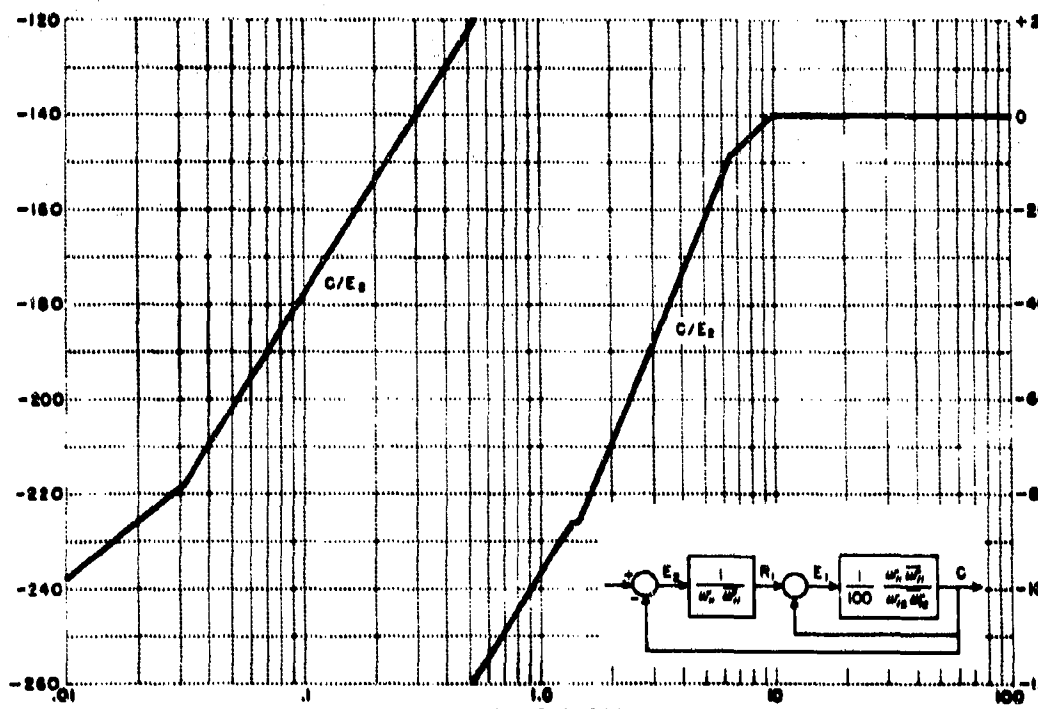


Figure 50. ω_{11}/ω_{12} , $q_1/r_1 = 1$

$$\left(\frac{s}{1.41}+1\right)\left[\left(\frac{s}{6.65}\right)^2+\frac{2(.603)}{6.65}s+1\right]\left(\frac{s}{9.447}+1\right) \qquad (9\text{-}99)$$

The agreement between the machine solution (Equation 9-99), found using all the entries of the F and G matrices, and the approximate root square locus answer of Equation 9-98 is quite good.

The model roots are defined by the denominator of Equation 9-49, with ℓ's replacing the f's. That is,

$$s(s-\ell_{33})\left[s^2-(\ell_{11}+\ell_{44})s+\ell_{11}\ell_{44}-\ell_{14}\right]$$

Substitution of the values given in Equation 9-47 yields:

$$s\left(\frac{s}{.026}+1\right)\left[\left(\frac{s}{1.166}\right)^2+\frac{2(.92)}{1.166}s+1\right]$$

The locations of the plant and model roots are sketched in Figure 51.

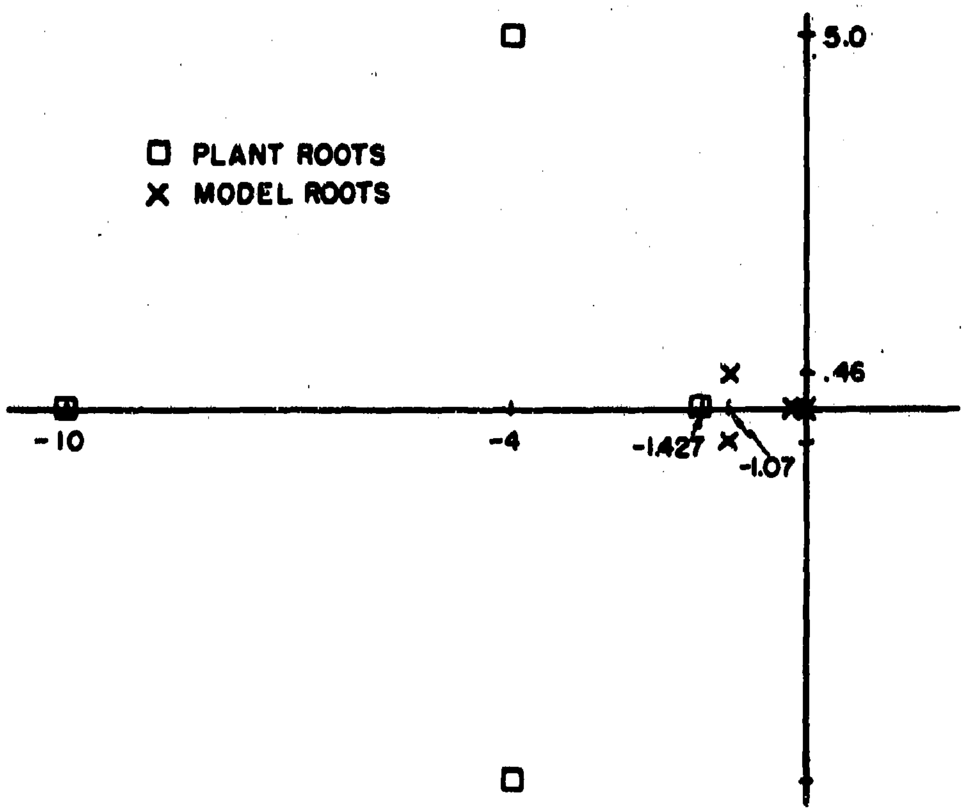


Figure 51. Plant and Model Root Locations in the Complex Plane

Since the preceding analysis has been quite lengthy, it is probably worthwhile to review the situation and see what has been accomplished up to this point.

Basically, we have investigated the closed-loop roots of the system as a function of the parameters of the performance index by using the equation

$$\left| R + G_p' \left[-Is - F_p'\right]^{-1} Q \left[Is - F_p\right]^{-1} G_p \right| = 0 \qquad (9\text{-}100)$$

This equation was extracted from the matrix model-following Wiener-Hopf equation (7-91). We have yet to solve Equation 7-91 for the optimal control or find the optimal outputs. In essence, the important remaining task is to find the zeros of the optimal control in order that the outputs of the system can be computed. Note that even though values of the q's and r's have been selected which give good closed-loop pole locations, the closed-loop zeros of the system can still force undesirable transient responses if we are unlucky.

In finding the optimal system outputs, two courses may be followed. One may elect to stay in the frequency domain and use the results of Section 7 or one may use the time domain computer program. Here we have elected to use the computer program. A brief summary of the results is given in the transient responses of Figures 52 and 53. The feedback gains required are tabulated in Table 4.

TABLE 4

GAINS SYNTHESIZED FROM DESIGN PROCEDURE

$\Delta\dot{\theta}_m$	$\Delta\theta_m$	ΔV_m	$\Delta\alpha_m$	$\Delta\dot{\theta}_p$	$\Delta\theta_p$	ΔV_p	$\Delta\alpha_p$	
-.073	3	-.978	-2.83	-.0823	-3.01	-.978	1.66	δ_T
5.14	39.2	-1.94	-23.3	-3.67	-38.7	1.94	19.3	δ_e
Feedforward Gains				Feedback Gains				

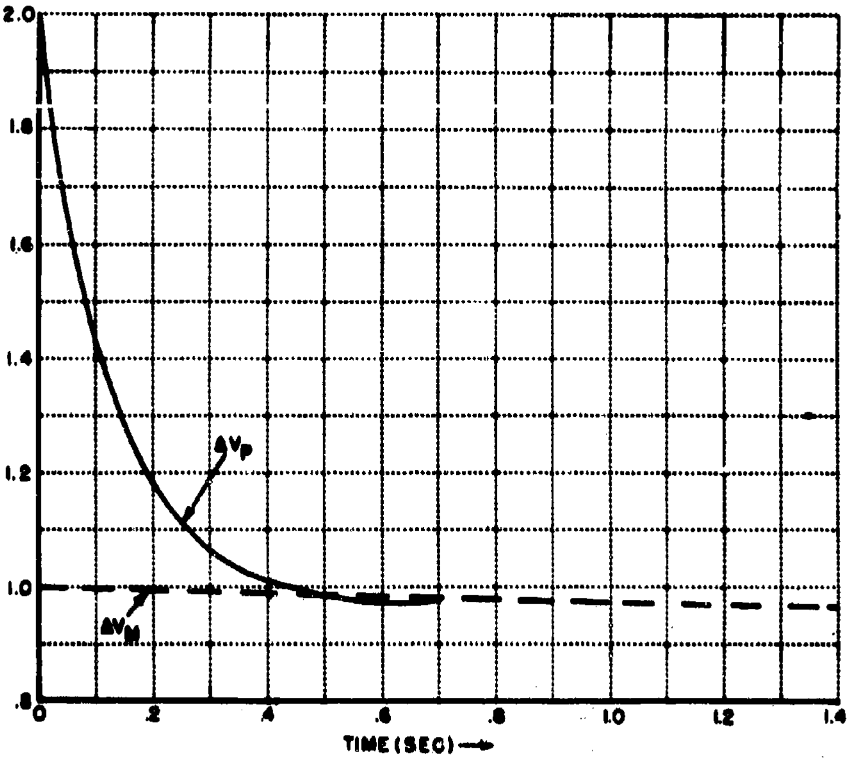


Figure 52. ΔV Response of Model and Plant

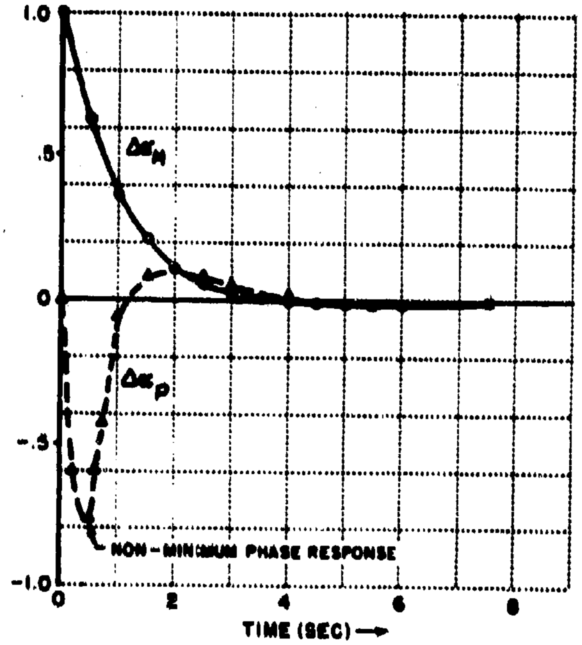


Figure 53. $\Delta\alpha$ Response of Model and Plant

From an inspection of these figures, one would probably conclude that while the model-following ability in ΔV is good, it is poor (in fact, non-minimum phase) in $\Delta\alpha$. Moreover, an inspection of the feedback gains shows the $\Delta\alpha_p$ to δ_e feedback of 19.3 deg/deg to be approximately twice as high as the maximum allowable.

If one considers this situation to be an unsatisfactory one, there is no recourse but to try again - that is, pick a set of q's and r's that will not only give desirable closed-loop poles, but also realizable feedback gains and closed-loop zeros that are conducive to acceptable transient responses. In this respect, our procedure is still a trial and error one - much the same as the classical Bode plot or root locus methods. However, note that one is now working with two controls, the elevator and the throttle. The effort expended on the optimal design has produced stable responses and clearly shows how to select the closed-loop poles. The basic difficulty that occurs with two controls is due to the fact that the closed-loop zeros of the system can also be varied. It is in failing to find the pattern of the closed-loop zeros as a function of the q's and r's that our analysis procedure breaks down. The reader is probably well aware of the enormous difficulties presented by a conventional trial and error design when two controllers are involved.

The example will be concluded at this point, since the intent was to demonstrate procedures rather than the execution of a successful design. Before closing, however, a comment is in order on the high gains which were encountered.

The high gains which result are primarily due to the design philosophy which requires that the phugoid roots of the plant be pushed out past the short period roots of the model. An alternate design philosophy, which would require smaller gains, is discussed briefly in the next section.

9.4 AN ALTERNATE DESIGN PHILOSOPHY FOR MODEL FOLLOWING

As seen in the previous section, excessively high feedback gains were required to force the open-loop plant roots to migrate far enough into the left-half plane so that the plant root closest to the origin was to the left of the model root farthest from the origin. This may have been expected, since a situation was being forced which required the low frequency phugoid roots of the plant to increase in frequency on the order of four or five decades. It can be argued that this design philosophy does not take advantage of the large (for most aircraft) separation between the phugoid and short period frequencies.

A design philosophy that would take advantage of this separation might be stated as follows:

1. Move the phugoid roots of the plant one or two octaves above those of the model phugoid
2. Move the short period roots of the plant one or two octaves beyond those of the model.

The situation is depicted in Figure 54.

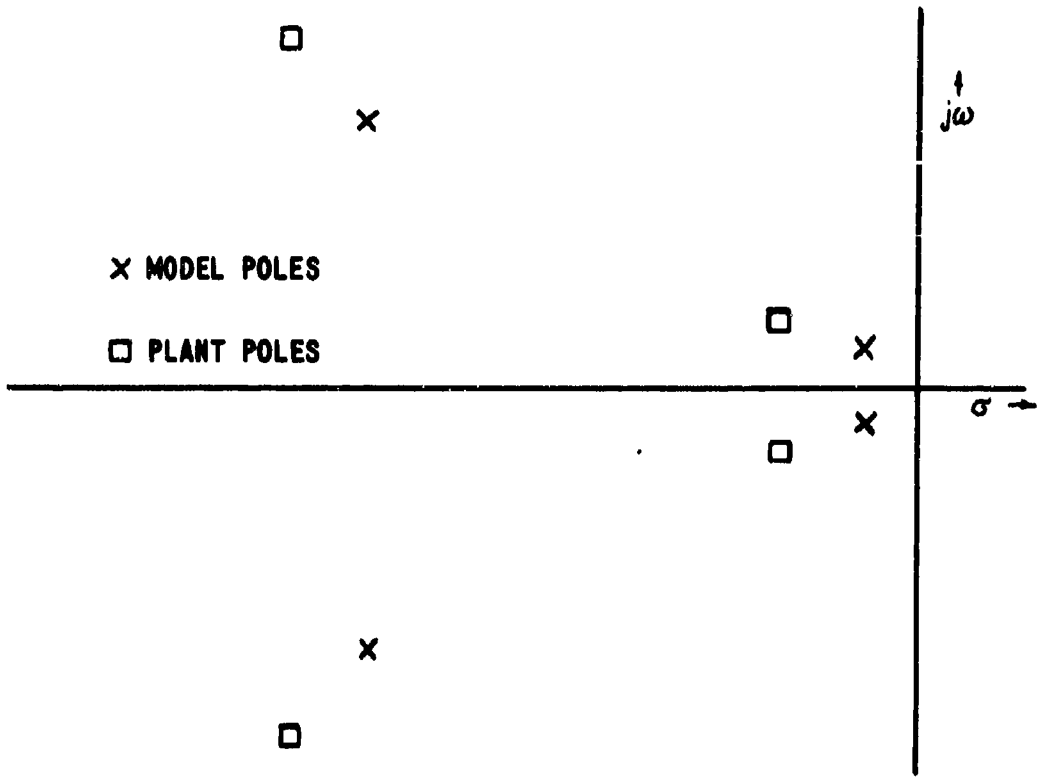


Figure 54. Plant and Model Roots for Alternate Design Philosophy

It can then be argued that the initial response will depend only on the short period roots of the model and plant with the model short period roots dominating. In a similar manner, the phugoid mode response will be dominated by the model phugoid roots.

It is fairly obvious that if "good" model following can be achieved with this philosophy, the resultant gains will be considerably lower than those required in the design effort of the previous section.

SECTION 10

CONCLUSIONS

1. Linear optimal control can be developed into an efficient and effective control system design tool. The characteristics of guaranteed stability, the possibility for a smooth, well-behaved response without excess control motion, and the probability of rapid machine synthesis for multi-controller, multi-output systems make the usage of the technique very attractive.

2. Linear optimal control does not have universal applicability. The fact that the performance index is of quadratic form a priori determines the type of distribution of the closed-loop poles of the system. The quadratic performance index defines the system function whose excess poles over zeros quickly tend toward a Butterworth distribution at the closed-loop natural frequency. For many applications, this approximation to a flat frequency response and a smooth, well-behaved transient response is desirable. For aircraft stability augmentation systems, with a human in outer loops, the quadratic criterion may not be the most desirable. It is felt, however, that a flight control system, properly designed by linear optimal techniques, would be acceptable to a pilot.

3. The time domain solutions to the problem and the frequency domain solutions are generally equivalent and complementary. The common link is the root square locus expression or analogous Bode plot technique. The multivariable root square locus expression, originally developed after minimization in the time domain, appears also in the frequency domain. This expression relates the parameters of the performance index to the closed-loop poles of the optimal system.

4. A model can be used to specify a dynamic response if other than a Butterworth approximation for the closed-loop system is desired. For a good approximation to the model, however, more than the root square locus expression for the closed-loop poles is needed. An expression, not obtained in this report, is needed for the closed-loop zeros of the multi-controller optimal system when the system is perturbed by a command input or a disturbance.

5. Linear optimal control system analysis is significantly simpler than conventional trial and error analysis. One root square locus per controller and per output is required regardless of the order of the system. In conventional design, a complete analysis requires an investigation of feedback from each state variable as well as to each controller from each output.

6. It appears that a wide variety of system excitations, including both command and disturbance inputs, can be easily included in the linear optimal control problem. In the time domain, it is a relatively simple matter to attach a vector describing a deterministic input to the plant description. In the frequency domain, any input that has a Laplace transform can be included in the problem formulation, including statistical inputs with well defined correlation functions. In any case, the regulator part of the optimal system is invariant under different inputs and can be separately analyzed.

7. The linear optimal control problem can be generalized to include a performance index involving any quadratic form of output and its time derivative along with the control and its time derivative.

8. Single controller linear optimal control problems are solvable simply and directly by using the root square locus to spectral factor a scalar expression.

9. This report describes a direct method for solving matrix Wiener-Hopf equations which does not require the factorization of a rational matrix. The technique is a general one which can be applied directly to Wiener-Hopf equations which arise in other branches of engineering and mathematics.

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APPENDIX I

DEVELOPMENT OF $H[Is-F]^{-1}G$ AS A MATRIX OF TRANSFER FUNCTIONS

The purpose of this appendix is to demonstrate that the matrix $H[Is-F]^{-1}G$ represents a matrix of transfer functions relating all the outputs of the system to the inputs. Consider the set of first-order equations of motion:

$$\dot{x} = Fx + Gu \qquad y = Hx \tag{I-1}$$

Taking the Laplace transform of Equation I-1 and solving for Y(s) yields

$$Y(s) = H\left[Is-F\right]^{-1} Gu(s) \qquad x(0) = 0 \tag{I-2}$$

It will be shown that the matrix $H[Is-F]^{-1}G$ is a weighting function or transfer function matrix defined by:

$$W(s) = \begin{bmatrix} \frac{y_1}{u_1}(s) & \frac{y_1}{u_2}(s) & \cdots & \frac{y_1}{u_p}(s) \\ \frac{y_2}{u_1}(s) & & & \cdot \\ \vdots & & & \cdot \\ & & & \cdot \\ \frac{y_n}{u_1}(s) & & & \frac{y_n}{u_p}(s) \end{bmatrix} = H\left[Is-F\right]^{-1}G \tag{I-3}$$

To show this, take the Laplace transform of Equation I-1 (initial conditions = 0)

$$\left[Is-F\right] x(s) = Gu(s) \qquad y(s) = Hx(s) \tag{I-4}$$

Using Cramer's Rule to solve for the individual transfer functions $y_i/u_j\,(s)$, and arranging the individual transfer functions as in Equations I-3, it is found that Equation I-3 and the matrix of transfer functions found by Cramer's Rule are the same. It may then be concluded that the "transfer function" matrix

$$W(s) = H\left[Is-F\right]^{-1}G$$

is a general matrix form of Cramer's Rule for finding transfer functions.

Consider, as an example, the following two-input, two-output second-order system expressed as two first-order differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{I-5}$$

A flow diagram of this system can be sketched as shown below:

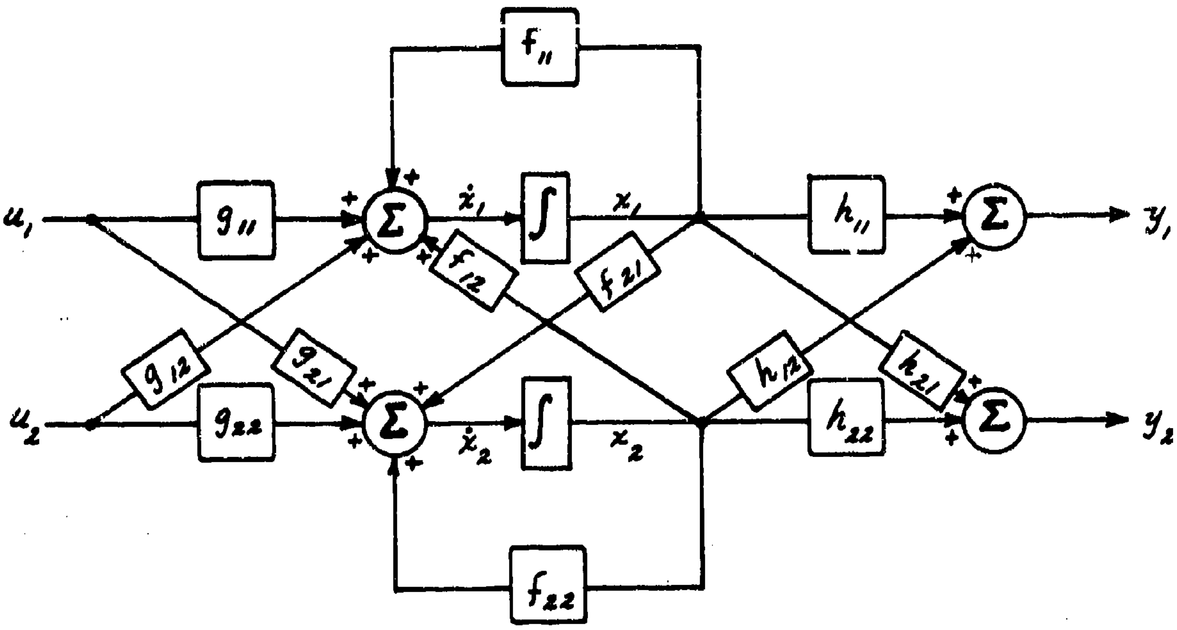

Figure I-1. Flow Diagram of
Two-Input, Two-Output System

By substituting for the integrators their Laplace transform equivalent $1/s$, and by using conventional signal flow techniques, the transfer function $x_i/u_j(s)$ can be determined by inspection:

$$\frac{x_1}{u_1}(s) = \frac{\frac{g_{11}}{s}\left(1-\frac{f_{22}}{s}\right)+\frac{g_{21}f_{12}}{s^2}}{1-\frac{f_{11}}{s}-\frac{f_{22}}{s}-\frac{f_{12}f_{21}}{s^2}+\frac{f_{11}f_{22}}{s^2}} = \frac{g_{11}s-g_{11}f_{22}+g_{21}f_{12}}{D(s)}$$

where $D(s) = s^2 - s(f_{11}+f_{22}) + f_{11}f_{22} - f_{12}f_{21}$

Similarly,

$$\frac{x_1}{u_2}(s) = \frac{g_{12}s+g_{22}f_{12}-g_{12}f_{22}}{D(s)}$$

$$\frac{x_2}{u_1}(s) = \frac{g_{21}s+g_{11}f_{21}-g_{21}f_{11}}{D(s)} \qquad \text{(I-6)}$$

$$\frac{x_2}{u_2}(s) = \frac{g_{22}s+g_{12}f_{21}-g_{22}f_{11}}{D(s)}$$

The transfer functions $y_i/u_j\,(s)$ then are, by inspection:

$$\left.\begin{aligned}
\frac{y_1}{u_1}(s) &= k_{11}\frac{x_1}{u_1}(s) + k_{12}\frac{x_2}{u_1}(s)\\
\frac{y_1}{u_2}(s) &= k_{11}\frac{x_1}{u_2}(s) + k_{12}\frac{x_2}{u_2}(s)\\
\frac{y_2}{u_1}(s) &= k_{21}\frac{x_1}{u_1}(s) + k_{22}\frac{x_2}{u_1}(s)\\
\frac{y_2}{u_2}(s) &= k_{21}\frac{x_1}{u_2}(s) + k_{22}\frac{x_2}{u_2}(s)
\end{aligned}\right\} \qquad \text{(I-7)}$$

To use Cramer's Rule, first find the Laplace transform of Equation I-5:

$$\begin{bmatrix} s-f_{11} & -f_{12} \\ -f_{21} & s-f_{22} \end{bmatrix}\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} \qquad \text{(I-8)}$$

The individual transfer functions then are:

$$\frac{x_1}{u_1}(s) = \frac{\begin{vmatrix} g_{11} & -f_{12} \\ g_{21} & s-f_{22} \end{vmatrix}}{\begin{vmatrix} s-f_{11} & -f_{12} \\ -f_{21} & s-f_{22} \end{vmatrix}} = \frac{g_{11}s - g_{11}f_{22} + g_{21}f_{12}}{D(s)} \qquad \text{(I-9)}$$

Also, solving for $x_1/u_2\,(s)$, $x_2/u_1\,(s)$, and $x_2/u_2\,(s)$ yields expressions identical to those of Equation I-6. Solving for $y_1/u_1\,(s)$, $y_2/u_1\,(s)$, $y_1/u_2\,(s)$, and $y_2/u_2\,(s)$ will then yield expressions identical to Equation I-7.

The same transfer functions can be found by using Equation I-10.

$$W(s) = H[Is-F]^{-1}G$$

$$= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} s-f_{22} & f_{12} \\ f_{21} & s-f_{11} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \frac{1}{D(s)}$$

$$= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} sg_{11} - f_{22}g_{11} + f_{12}g_{21} & sg_{12} - f_{22}g_{12} + f_{12}g_{22} \\ sg_{21} - f_{11}g_{21} + f_{21}g_{11} & sg_{12} - f_{11}g_{12} + f_{21}g_{12} \end{bmatrix} \frac{1}{D(s)}$$

$$= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \frac{x_1}{u_1}(s) & \frac{x_1}{u_2}(s) \\ \frac{x_2}{u_1}(s) & \frac{x_2}{u_2}(s) \end{bmatrix} = \begin{bmatrix} \frac{y_1}{u_1}(s) & \frac{y_1}{u_2}(s) \\ \frac{y_2}{u_1}(s) & \frac{y_2}{u_2}(s) \end{bmatrix} \qquad \text{(I-10)}$$

This development demonstrates that the same result is obtained by using either Cramer's Rule, signal flow techniques, or by direct matrix algebra.

APPENDIX II

DEVELOPMENT OF $\left|G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G\right|$ BY MINORS

The expression for the root square locus of multiple-input, multiple-output systems rapidly becomes complex as the number of controls becomes greater than one, although a fairly straightforward calculation based upon first- and second-order minors of the original system and input matrices will yield the proper root square locus expression. To show how this minor expansion comes about, consider the fourth-order, two input system described by the following matrices

$$[Is-F] = \begin{bmatrix} s+f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & s+f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & s+f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & s+f_{44} \end{bmatrix} \qquad G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \end{bmatrix} \tag{II-1}$$

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

The root square locus expression, as before, is given by:

$$\left| I + R^{-1}G'[-Is-F']^{-1}H'QH[Is-F]^{-1}G \right| = 0$$

or

$$\left| I + R^{-1}G' \frac{[-Is-F']^{adj}}{\bar{D}} H'QH \frac{[Is-F]^{adj}}{D} G \right| = 0 \tag{II-2}$$

where $[Is-F]^{adj}$ indicates the adjugate of a matrix; D is the determinant of the matrix $|Is-F|$.

The adjugate of the matrix $[Is-F]$ can be expressed in the following form:

$$[Is-F]^{adj} = \begin{bmatrix} A_{11} & -A_{21} & A_{31} & -A_{41} \\ -A_{12} & A_{22} & -A_{32} & A_{42} \\ A_{13} & -A_{23} & A_{33} & -A_{43} \\ -A_{14} & A_{24} & -A_{34} & A_{44} \end{bmatrix} \tag{II-3}$$

where A_{ij} are the first-order minors of the determinant $|Is - F|$ with the i^{th} row and j^{th} column deleted.

The matrix of transfer functions $H[Is-F]^{adj}G/D$ is given by

$$\frac{H[Is-F]^{adj}G}{D} = \frac{1}{D}\begin{bmatrix} A_{11} & -A_{21} & A_{31} & -A_{41} \\ -A_{12} & A_{22} & -A_{32} & A_{42} \\ A_{13} & -A_{23} & A_{33} & -A_{43} \\ -A_{14} & A_{24} & -A_{34} & A_{44} \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \end{bmatrix} \qquad H = I,\ 4\times 4$$

$$= \frac{1}{D}\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \\ N_{31} & N_{32} \\ N_{41} & N_{42} \end{bmatrix} \tag{II-4}$$

where N_{ij} is the numerator of the transfer function relating the i^{th} output to the j^{th} input.

The expression for the root square locus is given by, as before,

$$\left| I + R^{-1}G'[-Is-F]^{adj}H'QH[Is-F]^{adj}G \right| = 0 \tag{II-5}$$

or substituting, there results

$$\left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{D\bar{D}}\begin{bmatrix} r_1^{-1} & 0 \\ 0 & r_2^{-1} \end{bmatrix}\begin{bmatrix} \bar{N}_{11} & \bar{N}_{21} & \bar{N}_{31} & \bar{N}_{41} \\ \bar{N}_{12} & \bar{N}_{22} & \bar{N}_{32} & \bar{N}_{42} \end{bmatrix}\begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \\ N_{31} & N_{32} \\ N_{41} & N_{42} \end{bmatrix} \right| = 0 \tag{II-6}$$

where $\bar{N}_{ij}$ indicates the conjugate of N_{ij} with s replaced by -s.

To show its complexity, the above determinant expression is expanded below without using simplifying substitutions. The root square locus expression becomes:

$$
\begin{aligned}
&\Bigg\{ 1 + \frac{q_1}{r_1}\frac{N_{11}\bar{N}_{11}}{D\bar{D}} + \frac{q_2}{r_1}\frac{N_{21}\bar{N}_{21}}{D\bar{D}} + \frac{q_3}{r_1}\frac{N_{31}\bar{N}_{31}}{D\bar{D}} + \frac{q_4}{r_1}\frac{N_{41}\bar{N}_{41}}{D\bar{D}} + \frac{q_1}{r_2}\frac{N_{12}\bar{N}_{12}}{D\bar{D}} + \frac{q_2}{r_2}\frac{N_{22}\bar{N}_{22}}{D\bar{D}} + \frac{q_3}{r_2}\frac{N_{32}N_{32}}{D\bar{D}} \\
&+ \frac{q_4}{r_2}\frac{N_{42}\bar{N}_{42}}{D\bar{D}} + \frac{1}{r_1 r_2 (D\bar{D})^2}\Bigg[\begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} \begin{vmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{vmatrix} + \begin{vmatrix} q_1 & 0 \\ 0 & q_3 \end{vmatrix} \begin{vmatrix} N_{11} & N_{13} \\ N_{31} & N_{33} \end{vmatrix} \begin{vmatrix} \bar{N}_{11} & \bar{N}_{13} \\ \bar{N}_{31} & \bar{N}_{33} \end{vmatrix} \\
&+ \begin{vmatrix} q_1 & 0 \\ 0 & q_4 \end{vmatrix} \begin{vmatrix} N_{11} & N_{14} \\ N_{41} & N_{44} \end{vmatrix} \begin{vmatrix} \bar{N}_{11} & \bar{N}_{14} \\ \bar{N}_{41} & \bar{N}_{44} \end{vmatrix} + \begin{vmatrix} q_2 & 0 \\ 0 & q_3 \end{vmatrix} \begin{vmatrix} N_{21} & N_{22} \\ N_{31} & N_{33} \end{vmatrix} \begin{vmatrix} \bar{N}_{21} & \bar{N}_{22} \\ \bar{N}_{31} & \bar{N}_{33} \end{vmatrix} \\
&+ \begin{vmatrix} q_2 & 0 \\ 0 & q_4 \end{vmatrix} \begin{vmatrix} N_{21} & N_{22} \\ N_{41} & N_{42} \end{vmatrix} \begin{vmatrix} \bar{N}_{21} & \bar{N}_{22} \\ \bar{N}_{41} & \bar{N}_{42} \end{vmatrix} + \begin{vmatrix} q_3 & 0 \\ 0 & q_4 \end{vmatrix} \begin{vmatrix} N_{31} & N_{32} \\ N_{41} & N_{42} \end{vmatrix} \begin{vmatrix} \bar{N}_{31} & \bar{N}_{32} \\ \bar{N}_{41} & \bar{N}_{42} \end{vmatrix} \Bigg] \Bigg\}
\end{aligned}
\qquad \text{(II-7)}
$$

This expression is almost prohibitively complex, and would normally require computer operation, but the last six terms can be simplified considerably by showing that the determinants such as

$$\begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix}$$

can be expressed as second-order minors of the original system and input matrices.

From Equation II-4 the expression $N_{11}N_{22} - N_{12}N_{21}$ can be obtained as a function of the minors of the original system matrix.

$$
\begin{aligned}
N_{11} &= g_{11}A_{11} - g_{21}A_{21} + g_{31}A_{31} - g_{41}A_{41} \\
N_{12} &= g_{12}A_{11} - g_{22}A_{21} + g_{32}A_{31} - g_{42}A_{41} \\
N_{21} &= g_{11}A_{12} - g_{21}A_{22} + g_{31}A_{32} - g_{41}A_{42} \\
N_{22} &= -g_{12}A_{12} + g_{22}A_{22} - g_{31}A_{32} + g_{42}A_{42}
\end{aligned}
\qquad \text{(II-8)}
$$

$N_{11}N_{22} - N_{12}N_{21}$ becomes

$$
\begin{aligned}
&(g_{11}g_{22} - g_{12}g_{21})(A_{11}A_{22} - A_{12}A_{21}) + (g_{12}g_{31} - g_{11}g_{32})(A_{32}A_{11} - A_{31}A_{12}) \\
&+ (g_{11}g_{42} - g_{12}g_{41})(A_{11}A_{42} - A_{12}A_{41}) + (g_{21}g_{32} - g_{22}g_{31})(A_{21}A_{32} - A_{22}A_{31}) \\
&+ (g_{22}g_{41} - g_{12}g_{42})(A_{42}A_{21} - A_{41}A_{22}) + (g_{31}g_{42} - g_{32}g_{41})(A_{31}A_{42} - A_{32}A_{41})
\end{aligned}
\qquad \text{(II-9)}
$$

Using the determinant identity

$$AA_{ab,cd} = A_{ab}A_{cd} - A_{ad}A_{cb} \qquad \text{(II-10)}$$

where a, c are deleted rows of a determinant
b, d are deleted columns of a determinant

The resulting identities become

$$(g_{11} g_{22} - g_{12} g_{21})(A_{11}A_{22} - A_{12}A_{21}) = D(g_{11} g_{22} - g_{12} g_{21}) A_{11,22} \quad \text{a)}$$

$$(g_{12} g_{31} - g_{11} g_{32})(A_{32}A_{11} - A_{31}A_{12}) = D(g_{12} g_{31} - g_{11} g_{32}) A_{32,11} \quad \text{b)}$$

$$(g_{11} g_{42} - g_{12} g_{14})(A_{11}A_{42} - A_{12}A_{41}) = D(g_{11} g_{42} - g_{12} g_{14}) A_{11,42} \quad \text{c)}$$

$$(g_{21} g_{32} - g_{22} g_{31})(A_{21}A_{32} - A_{22}A_{31}) = D(g_{21} g_{32} - g_{22} g_{31}) A_{21,32} \quad \text{d)} \qquad \text{(II-11)}$$

$$(g_{22} g_{41} - g_{12} g_{42})(A_{42}A_{21} - A_{41}A_{22}) = D(g_{22} g_{41} - g_{12} g_{42}) A_{42,21} \quad \text{e)}$$

$$(g_{31} g_{42} - g_{32} g_{41})(A_{31}A_{42} - A_{32}A_{41}) = D(g_{31} g_{42} - g_{32} g_{41}) A_{31,42} \quad \text{f)}$$

where $A_{11,12}$ is a second ordered minor of the original F matrix. Specifically, for $A_{11,22}$ the result is

$$A_{11,22} = \begin{vmatrix} s+f_{33} & f_{34} \\ f_{43} & s+f_{44} \end{vmatrix} \qquad \text{(II-12)}$$

The six equations above, Equations II-11a through II-11f, can be expressed in terms of the original F matrix, with the G matrix substituted for the first two columns of the F matrix. Specifically, the result is

$$N_{11}N_{22} - N_{12}N_{21} = D \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & s+g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & s+g_{44} \end{vmatrix} \qquad \text{(II-13)}$$

and the term

$$\frac{1}{r_1 r_2 D^2 \bar{D}^2} \begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix} \begin{vmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{vmatrix}$$

is exactly equal to the expression

$$\frac{1}{r_1 r_2 D\bar{D}} \begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} \begin{vmatrix} g_{11} & g_{12} & f_{13} & f_{14} \\ g_{21} & g_{22} & f_{23} & f_{24} \\ g_{31} & g_{32} & s+f_{33} & f_{34} \\ g_{41} & g_{42} & f_{43} & s+f_{44} \end{vmatrix} \begin{vmatrix} g_{11} & g_{21} & g_{31} & g_{41} \\ g_{12} & g_{22} & g_{32} & g_{42} \\ f_{13} & f_{23} & -s+f_{33} & f_{43} \\ f_{14} & f_{24} & f_{34} & -s+f_{44} \end{vmatrix} \qquad \text{(II-14)}$$

Similarly, the other terms of the root square locus expression are of the same form as indicated above. For instance, the term

$$\frac{1}{r_1 r_2 D^2 \bar{D}^2} \begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} \begin{vmatrix} N_{11} & N_{12} \\ N_{31} & N_{32} \end{vmatrix} \begin{vmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{31} & \bar{N}_{32} \end{vmatrix}$$

$$= \frac{q_1 \; q_2}{r_1 r_2 D \; \bar{D}} \begin{vmatrix} g_{11} & f_{12} & g_{12} & f_{14} \\ g_{21} & s+f_{22} & g_{22} & f_{24} \\ g_{31} & f_{32} & g_{32} & f_{34} \\ g_{41} & f_{42} & g_{42} & s+f_{44} \end{vmatrix} \begin{vmatrix} g_{11} & g_{21} & g_{31} & g_{41} \\ f_{12} & -s+f_{22} & f_{32} & f_{42} \\ g_{12} & g_{22} & g_{32} & g_{42} \\ f_{14} & f_{24} & f_{34} & -s+f_{44} \end{vmatrix}$$

(II-15)

The basic importance of the equivalents given above, such as Equations II-14 and II-15, is the elimination of the exponent of the D^2 and $\bar{D}^2$ terms in the denominators of the expressions. It was assumed from the beginning of the multivariable root square locus expansion that a $D^2\bar{D}^2$ term could not exist. Therefore, $D\bar{D}$ had to be common to both the numerators and denominators of these expressions.

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13. ABSTRACT Linear optimal control theory has produced an important synthesis technique for the design of linear multivariable systems. In the present study, efficient design procedures, based on the general optimal theory, have been developed. These procedures make use of design techniques which are similar to the conventional methods of control system analysis. Specifically, a scalar expression is developed which relates the closed-loop poles of the multivariable optimal system to the weighting parameters of a quadratic performance index. Methods analogous to the root locus and Bode plot techniques are then developed for the systematic analysis of this expression. Examples using the aircraft longitudinal equations of motion to represent the object to be controlled are presented to illustrate design procedures which can be carried out in either the time or frequency domains. Both the model in the performance index and model-following concepts are employed in several of the examples to illustrate the model approach to optimal design.

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